Strictly Causal Functions Have a Unique Fixed Point

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Abstract

The denotational semantics of a deterministic timed system can be described by a function $F : (T \to V) \to (T \to V)$ with T partially ordered. The semantics of a feedback loop then is usually defined by a special (unique) fixed point of F but it is not always obvious that such a fixed point exists. This paper proves that every function F in the very general class of strictly causal functions has a unique fixed point.

Keywords: denotational semantics, timed system, strictly causal, unique fixed point

The denotational semantics of a deterministic timed system can be described by a function $F : (T \to V) \to (T \to V)$ where T is a partially (usually totally) ordered set representing time and $V \neq \emptyset$ is an arbitrary set of values [1, 7, 5, 4, 2, 3]. The behavior of a feedback loop is usually defined by some special (unique) fixed point of F.

If $T \subseteq \mathbb{R}$ then a metric d can be defined by $d(f_1, f_2) = \sup\{2^{-x} \mid x \in T \wedge f_1(x) \neq f_2(x)\}$ with $f_1, f_2 : T \to V$, $\sup(\emptyset) = -\infty$ and $2^{\infty} = \infty$ (with this definition f_1 and f_2 can have distance ∞). For $\varepsilon > 0$ the following holds: $d(f_1, f_2) \leq \varepsilon \Leftrightarrow (\forall x \in T : f_1(x) \neq f_2(x) \Rightarrow -\log_2(\varepsilon) \leq x)$.

$((T \to V), d)$ is a complete ultra-metric space. Banach's fixed point theorem ensures the existence of a unique fixed point for F if it exists $\delta < 1$ with

$$d(F(f_1), F(f_2)) < \delta * d(f_1, f_2) \text{ for } f_1 \neq f_2 \tag{C1}$$

— i.e. F is δ causal. [1, 8, 4, 2, 3] are based on this result. [7] proves the same result not using metric spaces. [5] shows the existence of a unique fixed point if for all $x \in T$ there exists a $\delta(x) < 1$ with $d(F(f_1), F(f_2)) \leq \delta(x) * 2^{-x}$ if $d(f_1, f_2) \leq 2^{-x}$ also without using metric spaces. Certain restrictions to T apply.

(C1) is a rather severe restriction for F and there exist programming languages e.g. VHDL [6] which are able to describe feedback loops which violate this condition.

A natural relaxation of condition (C1) is

$$d(F(f_1), F(f_2)) < d(f_1, f_2) \text{ for } f_1 \neq f_2 \tag{C2}$$

— i.e. F is strictly causal [3]. It is easy to see that F then has at most one fixed point but there is no metric space theorem that ensures the existence of a fixed point of F. In [3] it remains open if (C2) ensures the existence of a fixed point.

This definition of strictly causal still restricts F:

(1) an intuitively strictly causal function like $F_N : ([0,1] \to \{0,1\}) \to ([0,1] \to \{0,1\})$ with $F_N(f)(0) = 0$ and $F_N(f)(x) = f(0)$ for $x \in]0,1]$ does not satisfy (C2),

(2) T has to be a subset of $\mathbb{R}$ though other totally ordered sets like $\mathbb{R} \times \mathbb{N}$ with lexicographical ordering might be more natural in some circumstances (as it is the case for VHDL ([6] p.22-25), where so called delta delay operations exist which do not have any delay regarding the real time component but which still enforce an ordering of these operations), and

(3) T must be totally ordered, though an only partially ordered set T could be used to model objects which are only sometimes synchronized. E.g. $T_U = \{(x,y) \in \mathbb{R}^2 \mid \exists n \in \mathbb{Z} : n \leq x < n+1 \wedge n \leq y < n+1\}$ with $(x_1, y_1) \leq (x_2, y_2) :\Leftrightarrow x_1 \leq x_2 \wedge y_1 \leq y_2$ could be used to model 2 objects with independent clocks which synchronize at all $n \in \mathbb{Z}$.

This article therefore defines F to be strictly causal if

$$\mathit{Head}(f_1, f_2) \subsetneq \mathit{Head}(F(f_1), F(f_2)) \text{ for } f_1 \neq f_2 \tag{C3}$$

with

$$\mathit{Head}(f_1, f_2) = \{y \in T \mid \forall x \in T \text{ with } x \leq y : f_1(x) = f_2(x)\}$$

being the common prefix set of f_1 and f_2. (C3) is a generalization of (C2) which avoids these drawbacks of (C2).

Another approach is to have $\hat{F} : X \to X$ where $X = \{f : I \to V \mid I \in R\}$ with $R = \{I \subseteq T \mid \forall y \in I \, \forall x \in T : (x \leq y \Rightarrow x \in I)\}$ is the cpo of partial functions with a start interval as domain and $\subseteq$ as partial ordering. If $\hat{F}$ is monotone then the theorem from Knaster-Tarski implies that $\hat{F}$ has a least fixed point.
For a given $F : (T \to V) \to (T \to V)$ satisfying (C2) or (C3) there always exists a monotone $\hat{F} : X \to X$ with $F(f) = \hat{F}(f)$ for $f \in (T \to V)$.
Unfortunately, $\hat{F}$ can have more than one fixed point and the least fixed point is not necessarily total. For some F satisfying (C2) or (C3) there does not exist any monotone $\hat{F}$ with a total least fixed point.
A (contrived) example is $F_M : ([0,1] \to \mathbb{N})^2$, $F_M(f)(x) = 0$ if $x \leq r(f)$ and $F_M(f)(x) = 1$ otherwise, where $r(f) = 1/(t(f)+1)$ if $t(f) < \infty$ and $r(f) = 1$ otherwise with $t(f) = \inf\{\sup\{f(y) \mid y < z\} \mid z \in \,]0,1]\}$. F_M satisfies (C2) and (C3) but not (C1). Any monotone $\hat{F}_M$ with $\hat{F}_{M|[0,1]\to\mathbb{N}} = F_M$ has $f_M : [0] \to \mathbb{N}$, $f_M(0) = 0$ as a fixed point.
This is a problem with semantics that only define total functions as behavior.

In this article it is shown that condition (C3) implies the existence of a unique fixed point. Then a worst case estimation on how the fixed point changes when F changes is given. As an application of the first result the problem if condition (C2) implies the existence of a unique fixed point is answered positively.

Theorem 1. *Let $(T, \leq)$ be a partially ordered set, $V \neq \emptyset$ an arbitrary set and $Z = (T \to V)$. Let $F : Z \to Z$ be strictly causal according to (C3).*
Then F has a unique fixed point, i.e. it exists exactly one $f_0 \in Z$ with $F(f_0) = f_0$.
In addition the following holds: $\forall \hat{f} \in Z : Head(\hat{f}, F(\hat{f})) = Head(\hat{f}, f_0)$.

Proof: Let $R = \{I \subseteq T \mid \forall y \in I \, \forall x \in T : (x \leq y \Rightarrow x \in I)\}$ be the set of all prefix sets of T and $X = \{f : I \to V \mid I \in R\} \supset Z$. X is a cpo with the following relation as partial order:
$\forall f_1 : I_1 \to V, f_2 : I_2 \to V : f_1 \sqsubseteq f_2 :\Leftrightarrow (I_1 \subseteq I_2 \wedge \forall x \in I_1 : f_1(x) = f_2(x))$.
$Head(,)$ can be extended to $Head(f_1, f_2) = \{y \in I_1 \cap I_2 \mid \forall x \in T \text{ with } x \leq y : f_1(x) = f_2(x)\}$ with $f_1 : I_1 \to V, f_2 : I_2 \to V \in X$.
Assumption (C3) implies:
$\forall f \in X, f_1, f_2 \in Z \subseteq X$ with $f \sqsubset f_1$ and $f \sqsubset f_2$ and $f \sqsubset F(f_1)$ there exists $g \in X$ with $f \sqsubset g$ and $g \sqsubseteq F(f_1)$ and $g \sqsubseteq F(f_2)$. (*)

Take $v_0 \in V$ then F can be extended to a function $H : X \to Z$ in the following way: For $(f : I \to V) \in X$ let $G(f)$ denote the function
$G(f) : T \to V$, $\forall x \in I : G(f)(x) = f(x)$ and $\forall x \in T \setminus I : G(f)(x) = v_0$.
Then for $f \in X$ define $H(f) = F(G(f))$.
Note that for all $f \in X$ holds: $f \sqsubseteq G(f)$.
Put $\hat{f} \in Z$ arbitrarily and let $\hat{f}_0 : \mathit{Head}(\hat{f}, F(\hat{f})) \to V$, $x \mapsto \hat{f}(x)$.

Let $P = \{f \in X \mid \hat{f}_0 \sqsubseteq f \sqsubset H(f)\}$ and $Q = \{f \in X \mid \hat{f}_0 \sqsubseteq f = H(f)\}$.
In the following it will be shown that $Q \neq \emptyset$:
(1) *$P \cup Q$ is not empty:*
Obviously $\hat{f}_0 \in P \cup Q$.
(2) Define $C = \{c \subseteq P \cup Q \mid \emptyset \neq c \text{ is totally ordered}\}$ the set of all non-empty chains in $P \cup Q$. In the following it is shown that C is inductively ordered with respect to $\subseteq$:
(a) $C \neq \emptyset$ since $P \cup Q \neq \emptyset$.
(b) Let $B \subseteq C$ be a chain of chains in $P \cup Q$. Then $\bigcup_{c \in B} c$ is a chain in C and is an upper bound of B.
(3) Zorn's Lemma now implies the existence of a maximal chain $c_0 \in C$.
(4) *$g_0 := \bigsqcup c_0 \in P \cup Q$:*
g_0 exists because X is a cpo and $c_0 \neq \emptyset$ is a chain. Take any $f \in c_0$ with $f \neq g_0$, thus $f \sqsubset g_0$, then (*) with $f_1 = G(f)$ and $f_2 = G(g_0)$ ensures that there exists $h \in X$ with $f \sqsubset h$ and $h \sqsubseteq F(G(f))$ and $h \sqsubseteq F(G(g_0)) = H(g_0)$, i.e. $f \sqsubset H(g_0)$. This implies that $g_0 \sqsubseteq H(g_0)$ and hence $g_0 \in P \cup Q$.
(5) *If $g_0 \in P$ then there exists $h_0 \in P \cup Q : g_0 \sqsubset h_0$:*
(*) with $f_1 = G(g_0)$ and $f_2 = H(g_0)$ ensures that there exists $h_0 \in X$ with $g_0 \sqsubset h_0$ and $h_0 \sqsubseteq F(G(g_0)) = H(g_0)$ and $h_0 \sqsubseteq F(H(g_0))$.
If $h_0 = H(g_0)$ then $h_0 = F(H(g_0))$, i.e. $h_0 \in Q$ and $g_0 \sqsubset h_0$.
If $h_0 \neq H(h_0)$, i.e. $h_0 \sqsubset H(g_0)$ then $h_0 \sqsubset G(h_0)$ and (*) with $f_1 = H(g_0)$ and $f_2 = G(h_0)$ implies that there exists $\tilde{h} \in X$ with $h_0 \sqsubset \tilde{h}$ and $\tilde{h} \sqsubseteq F(H(g_0))$ and $\tilde{h} \sqsubseteq F(G(h_0)) = H(h_0)$, i.e. $h_0 \sqsubset H(h_0)$, and hence $h_0 \in P$ and $g_0 \sqsubset h_0$.
(6) Assume for the sake of contradiction $g_0 \in P$. Then $c_0 \cup \{h_0\} \in C$ really includes c_0 which is a contradiction to the maximality of c_0 in C. Hence $g_0 \in Q$ and thus $Q \neq \emptyset$.
Obviously $\hat{f}_0 \sqsubseteq g_0$, i.e. $\mathit{Head}(\hat{f}, F(\hat{f})) \subseteq \mathit{Head}(\hat{f}, g_0)$.
Since $\mathit{Head}(\hat{f}, g_0) \cap \mathit{Head}(F(\hat{f}), g_0) \subseteq \mathit{Head}(\hat{f}, F(\hat{f}))$ and $\mathit{Head}(\hat{f}, g_0) \subseteq \mathit{Head}(F(\hat{f}), H(g_0)) = \mathit{Head}(F(\hat{f}), g_0)$ it is $\mathit{Head}(\hat{f}, g_0) \subseteq \mathit{Head}(\hat{f}, F(\hat{f}))$.
Together this implies $\mathit{Head}(\hat{f}, g_0) = \mathit{Head}(\hat{f}, F(\hat{f}))$.

Every fixed point of H is a maximal element in X because $H(f) \in Z$ for all $f \in X$ and Z is the set of maximal elements in X.
Now consider $f_1 \in Q$ and $f_2 \in Z$ with $f_1 \neq f_2$. Then (C3) implies that there exists $x_0 \in \mathit{Head}(H(f_1), H(f_2)) \setminus \mathit{Head}(f_1, f_2)$ with $H(f_2)(x_0) = H(f_1)(x_0) = f_1(x_0) \neq f_2(x_0)$, i.e. $f_2 \neq H(f_2)$ and therefore it exists a unique fixed point f_0.
Because f_0 is maximal in X, f_0 is a fixed point of F, too. Since every fixed point of F is a fixed point of H, F does not have more than one fixed point. Since $\hat{f}$ was chosen arbitrarily and since the fixed point is unique the addition of the theorem holds. □

The proof of theorem 1 uses Zorn's lemma which is equivalent to the axiom of choice and hence is inherently non-constructive.
However, step (5) in the proof of theorem 1 can be used to compute a new approximation of the fixed point when given an approximation.
Step (4) shows that the limit of a chain of approximations is an approximation of the fixed point, i.e. is either the fixed point itself or can be used as the starting point for further computations of approximations.
The definition of P, Q shows how a first approximation can be chosen ($f : \emptyset \to V$ can also be taken).
On a real computer usually at most the first limit can be computed. On the other hand the first limit can be the fixed point even if F is not δ-causal. E.g. F_N is not δ-causal, does even not satisfy (C2), but the fixed point is easy to compute.

As an application of the addition of this theorem it is shown that slight changes in F imply only slight changes in the fixed point of F.

Lemma 1. $(T, \leq)$ *be a partially ordered set,* $V \neq \emptyset$ *an arbitrary set and* $Z = (T \to V)$. *Let* $F_1, F_2 : Z \to Z$ *be strictly causal according to (C3) with unique fixed points* f_1, f_2 *respectively. Consider* $S \subseteq T$ *such that for all* $f \in Z$: $S \subseteq \mathit{Head}(F_1(f), F_2(f))$.
Then it holds: $S \subseteq \mathit{Head}(f_1, f_2)$.

Proof: The addition of theorem 1 provides $\mathit{Head}(f_1, F_2(f_1)) = \mathit{Head}(f_1, f_2)$ and due to the assumption of the lemma
$S \subseteq \mathit{Head}(F_1(f_1), F_2(f_1)) = \mathit{Head}(f_1, F_2(f_1))$.
Together this implies $S \subseteq \mathit{Head}(f_1, f_2)$. □

In the literature condition (C2) is used to define strictly causal, (C2) is e.g. used in [3] on page 28f. There it is left open if (C2) ensures the existence of a fixed point. In the following the result that (C2) implies the existence of a unique fixed point is therefore proven as an application of theorem 1.

Theorem 2. *Let $T \subseteq \mathbb{R}$, $V \neq \emptyset$ and $Z = (T \to V)$. For $f_1, f_2 \in Z$ with $f_1 \neq f_2$ let $d(f_1, f_2) = \sup\{2^{-x} \mid x \in T \text{ with } f_1(x) \neq f_2(x)\}$ and $d(f_1, f_1) = 0$. If for a function $F : Z \to Z$ for all $f_1, f_2 \in Z$ with $f_1 \neq f_2$ holds that $d(F(f_1), F(f_2)) < d(f_1, f_2)$, i.e. F is strictly causal according to (C2), then F has a unique fixed point f_0.*

In addition the following holds: $\forall \hat{f} \in Z : d(\hat{f}, F(\hat{f})) = d(\hat{f}, f_0)$

Proof: Let $f_1, f_2 \in Z$ with $f_1 \neq f_2$. Obviously $d(F(f_1), F(f_2)) < d(f_1, f_2)$ implies that $Head(f_1, f_2) \subsetneq Head(F(f_1), F(f_2))$. Then theorem 1 guarantees that F has a unique fixed point f_0.
The addition of theorem 1 ensures that the addition of this theorem holds.
□

In general it is much easier to prove that condition (C2) or (C3) hold than to prove that condition (C1) is satisfied, because (C2) and (C3) are local conditions whereas (C1) is global.
Moreover there exist cases where (C1) is not satisfied and hence cannot be applied, but (C2) or (C3) are, as it is the case in [3].
Overall the results in this article make it much easier to ensure that a denotational semantics is always defined than with using metric space arguments. It is also easier than an application of Knaster-Tarski if the semantics is required to be a total function.
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