An operator representation for weighted spaces of vector valued holomorphic functions

Klaus D. Bierstedt and Silke Holtmanns

ABSTRACT. *For any weighted space $HV(G)$ of holomorphic functions on an open set $G \subset \mathbb{C}^N$ with a topology stronger than that of uniform convergence on the compact sets and for any quasibarrelled space E we prove the topological isomorphism $HV(G, E_b') = \mathcal{L}_b(E, HV(G))$ and derive a similar, more complicated isomorphism for weighted spaces of continuous functions. This generalizes results of [3], [7] and [6] and should be compared with the ε-product representations for the corresponding spaces of functions with o-growth conditions. At the end we also show the topological isomorphism $HV_1(G_1, HV_2(G_2)) = H(V_1 \otimes V_2)(G_1 \times G_2)$.*

Mathematics Subject Classification: primary 46E40, secondary 30H05, 32A37, 46A04, 46A08, 46A20, 46A32, 46E10.

1. Introduction

Weighted spaces $CV_0(X)$ and $CV_0(X, E)$ of scalar and vector valued continuous functions with o-growth conditions, given by a Nachbin family V on a completely regular Hausdorff space X, were introduced by Leopoldo Nachbin in his work on approximation theory (see [17]). The framework of weighted spaces has proved to be both versatile enough to cover virtually all examples arising in various branches of analysis and yet restrictive enough to permit important theorems. In the vector valued case, the first-named author studied spaces of type $CV_0(X, E)$ in connection with ε-product and (completed) ε-tensor product representations

$$CV_0(X, E) = CV_0(X)\varepsilon E = \mathcal{L}_e(E_c', CV_0(X)) = CV_0(X)\check{\otimes}_\varepsilon E$$

and the approximation property (cf. [1], [2]). Here E_c' denotes the dual of the (complete) locally convex space E, equipped with the topology of uniform convergence on the compact subsets of E, and the space $\mathcal{L}_e(E_c', CV_0(X))$ of all continuous linear operators from E_c' into $CV_0(X)$ is endowed with the topology of uniform convergence on the equicontinuous subsets of E'. Such representations are very helpful since they reduce many problems for spaces of vector valued continuous functions to the corresponding spaces of scalar functions and to permanence properties of the ε-(tensor) product.

Weighted spaces of continuous functions have always been a handy tool in the treatment of the corresponding spaces $HV_0(G)$ and $HV_0(G, E)$ of holomorphic functions on open subsets G of $\mathbb{C}^N$, $N \geq 1$, which are much more important from the point of view of applications in analysis (e.g., see [9]). As in the case of continuous functions, also $HV_0(G, E)$ has an ε-product representation (see [1])

$$HV_0(G, E) = HV_0(G)\varepsilon E = \mathcal{L}_e(E_c', HV_0(G)).$$

In this article we will mainly be interested in the spaces $CV(X,E)$ and $HV(G,E)$ of continuous and holomorphic functions with O-growth conditions, and these spaces do not have ε-product representations in general. In fact, it was proved in [1] that

$$CV(X)\varepsilon E = CV^p(X,E), \;\; HV(G)\varepsilon E = HV^p(G,E)$$

holds, where $CV^p(X,E)$ denotes the space of all continuous E-valued functions f on X for which $(vf)(X)$ is precompact in E for each $v \in V$ and where $HV^p(G,E)$ is defined in an analogous way. If one wants to represent the whole space $CV(X,E)$ or $HV(G,E)$ as a space of operators, then one clearly has to consider operators from the strong dual E'_b into $CV(X)$ or $HV(G)$, respectively. But at this point one immediately encounters difficulties with the bidual (which did not appear in the previous case since the topology of E'_c is compatible with the duality of E and E' when E is quasicomplete). Analyzing this fact, it turns out that the space of operators which one gets by representing $CV(X,E)$ or $HV(G,E)$ is rather complicated (Proposition 4). That is why we prefer to start in Proposition 1 with an operator representation of $CV(X,E'_b)$, where E is a quasibarrelled locally convex space (so that E'_b is quasicomplete). Even this representation – while being more easily accessible than the one in Proposition 4 – is hard to apply in general. However, it is nice and canonical for sequence spaces and in the important case of spaces of holomorphic functions, where we obtain the topological isomorphism (Theorem 2)

$$HV(G,E'_b) = \mathcal{L}_b(E, HV(G)).$$

In Corollary 3, it is pointed out that such an isomorphism also holds for spaces of harmonic functions and for function spaces of a similar type, see [8]. Next, under rather general conditions, the space $HV(G)$ has a predual P (see [4]) so that the operator representation can then be written $HV(G,E'_b) = \mathcal{L}_b(P,E'_b)$ (Corollary 5).

In a similar spirit, ε-(tensor) product representations can also be used to reduce certain problems for spaces of functions in several variables on product sets to the corresponding spaces of functions in one variable. E.g., in [1] it was proved that, under mild assumptions, the following topological isomorphisms hold:

$$\begin{aligned} C(V_1 \otimes V_2)_0(X_1 \times X_2) &= C(V_1)_0(X_1)\varepsilon C(V_2)_0(X_2) = C(V_1)_0(X_1, C(V_2)_0(X_2)), \\ H(V_1 \otimes V_2)_0(G_1 \times G_2) &= H(V_1)_0(G_1)\varepsilon H(V_2)_0(G_2) = H(V_1)_0(G_1, H(V_2)_0(G_2)). \end{aligned}$$

While it is well-known that the first of these isomorphisms does not carry over from o- to O-growth conditions, it may come as a surprise to realize that

$$H(V_1 \otimes V_2)(G_1 \times G_2) = HV_1(G_1, HV_2(G_2))$$

holds in full generality (Theorem 7).

Finally, we would like to point out that the proofs of all the results in this article are not too hard, once one is familiar with the duality theory of locally convex spaces – in particular, with the notion of equicontinuity –, with Montel's and Hartogs' theorems, and with the fact that it is enough to test the weak holomorphy of a vector valued function on the elements of a separating set in the dual (e.g., cf. [12]).

Notation. Let X be a completely regular Hausdorff space. A *weight* v is a non-negative upper semicontinuous function on X. A *Nachbin family* V on X is a system of weights with the property that for $v_1, v_2 \in V$ and $\lambda > 0$ there is $v \in V$ with $\lambda v_1, \lambda v_2 \leq v$. We will always require that $V > 0$; that is, for each $x \in X$ there exists $v \in V$ with $v(x) > 0$. $V \geq W$ means that for each compact set K in X there is $v \in V$ with $\inf_K v > 0$; this clearly holds whenever V contains a continuous (strictly) positive function. For a locally convex space E, $cs(E)$ stands for the system of all continuous seminorms on E, and for $p \in cs(E)$, B_p denotes the closed unit ball of p. The topology of the strong dual E'_b of E is given by the system $\{p_B\}_B$ of seminorms, where B runs through all (closed absolutely convex) bounded subsets of E and $p_B(e') := \sup_{b \in B} |e'(b)|$ for $e' \in E'$. Letting $C(X, E)$ (and $CB(X, E)$) denote the space of all continuous (respectively, all bounded and continuous) functions on X with values in E, the *weighted spaces of continuous E-valued functions* are defined by

$$\begin{aligned} CV(X, E) &:= \{f \in C(X, E);\ (vf)(X) \text{ bounded in } E \ \ \forall v \in V\}, \\ CV_0(X, E) &:= \{f \in C(X, E);\ vf \text{ vanishes at infinity on } X \ \ \forall v \in V\}, \end{aligned}$$

endowed with the weighted (locally convex) topology given by the seminorms

$$q_{v,p}(f) := \sup_{x \in X} v(x) p(f(x)),\ \ v \in V,\ \ p \in cs(E).$$

Since $V > 0$, the weighted topology is Hausdorff, and if $V \geq W$, then the weighted topology is stronger than the compact-open topology τ_0 (i.e., the topology of uniform convergence on the compact subsets of X). In the latter case, $CV(X, E)$ is complete for complete E whenever X is a *$k_{\mathbb{R}}$-space*, which means that a real valued function g on X must be continuous if all its restrictions to the compact subsets of X are continuous. In case $E = \mathbb{C}$, E is omitted from the notation, and the spaces are denoted by $CV(X)$ and $CV_0(X)$, respectively. The canonical seminorms will then be called $p_v := q_{v,|.|}$, and B_v denotes the closed unit ball of p_v. If V consists of the positive multiples of a single positive weight v, one writes $Cv(X, E)$ and $Cv_0(X, E)$ etc.

Turning to the case of holomorphic functions, let G be an open subset of $\mathbb{C}^N$, V a Nachbin family on G, and let E be a quasicomplete locally convex space. As usual, a function $f : G \to E$ is said to be *holomorphic* if it is weakly holomorphic; that is, $e' \circ f$ is holomorphic for each $e' \in E'$. $H(G, E)$ denotes the space of all holomorphic functions from G into E. Now the *weighted spaces of vector valued holomorphic functions* are given by

$$HV(G, E) := CV(G, E) \cap H(G, E), HV_0(G, E) := CV_0(G, E) \cap H(G, E),$$

where the topology is the one induced by $CV(G, E)$. Again, the weighted spaces of scalar holomorphic functions are denoted by $HV(G)$ and $HV_0(G)$, and the notations $Hv(G, E)$, $Hv_0(G, E)$ etc. are clear.

One starting point for the present line of research was the topological isomorphism

$$l_\infty(I, E'_b) = \mathcal{L}_b(E, l_\infty(I)),$$

which holds for quasibarrelled spaces E (see S. Dierolf [11], proof of Proposition 5.14). Later on, this topological isomorphism was extended to vector valued co-echelon spaces as follows:

$$K_\infty(E'_b) = K_\infty(I, \overline{V}; E'_b) = \mathcal{L}_b(E, K_\infty(I, \overline{V})) = \mathcal{L}_b(E, K_\infty),$$

where E must again be quasibarrelled, see [3], I, Proposition 2.13.(a). When the elements of the co-echelon space are written as functions, the canonical isomorphism I of $K_\infty(E_b')$ onto $\mathcal{L}_b(E, K_\infty)$ has the form

$$f \longrightarrow (e \to (i \to [f(i)](e))), \ e \in E, \ i \in I,$$

while its inverse $J = I^{-1}$ is given by $T \longrightarrow (i \to \delta_i \circ T)$, where δ_i denotes evaluation at i; that is, $[J(T)](i)(e) = [T(e)](i)$ for $e \in E$, $i \in I$ and $T \in \mathcal{L}(E, K_\infty)$.

An isomorphism similar to $l_\infty(I, E_b') = \mathcal{L}_b(E, l_\infty(I))$ was also considered in the context of spaces of bounded continuous functions in [7], Appendix to Section 2. It turned out that if E is quasibarrelled, then $CB(X, E_b')$ is topologically isomorphic to a *subspace* of $\mathcal{L}_b(E, CB(X))$, but that the mapping corresponding to I above is not surjective in general (e.g., it was shown in [7] that it is not surjective for $E = CB(X)$, $X = [0, 1]$). However, [7], Proposition B showed that, in fact,

$$H^\infty(G, E_b') = \mathcal{L}_b(E, H^\infty(G))$$

holds topologically for any quasicomplete (quasi-) barrelled space E. We will show below that this precise behavior extends from $CB(X)$ and $H^\infty(G)$ to $CV(X)$ and $HV(G)$, respectively.

2. Operator representations for spaces of vector valued functions

In our first result, the space $\mathcal{L}_b(E, CV(X))$ of all continuous linear operators from E into $CV(X)$, endowed with the topology of uniform convergence on the bounded subsets of E, appears. Note that the system of seminorms $\{q_{B,v};\ B$ bounded in $E,\ v \in V\}$ induces the topology of this space, where

$$q_{B,v}(T) := \sup_{b \in B} p_v(T(b)) = \sup_{b \in B} \sup_{x \in X} v(x)|[T(b)](x)|, \ T \in \mathcal{L}(E, CV(X)).$$

1. Proposition. *If E is a quasibarrelled locally convex space, $CV(X, E_b')$ is topologically isomorphic to the following topological subspace of $\mathcal{L}_b(E, CV(X))$:*

$$\mathcal{LE}(E, CV(X)) := \{T \in \mathcal{L}(E, CV(X));\ T(B) \text{ equicontinuous on } X\ \forall B \text{ bounded } \subset E\}.$$

The canonical isomorphism I of $CV(X, E_b')$ into $\mathcal{L}_b(E, CV(X))$ is given by

$$[I(f)](e) : x \longrightarrow [f(x)](e), \ e \in E, \ f \in CV(X, E_b').$$

PROOF. $[I(f)](e)$ is a well-defined map from X into $\mathbb{C}$ (the scalar field of E), which clearly is continuous on X as the composition of f with the evaluation at e. Fix $v \in V$. Then $(vf)(X)$ is bounded in E_b', hence a fortiori bounded in $E'[\sigma(E', E)]$. Thus, for each $e \in E$, $(v[I(f)])(e)(X) = (vf)(X)(e)$ must be bounded; that is, $[I(f)](e) \in CV(X)$ for each $e \in E$. We have proved that $I(f)$ is a (linear) mapping from E into $CV(X)$.

Moreover, if $f \in C(X, E_b')$, then, clearly, $x \to [f(x)](b)$ describes an equicontinuous set of scalar functions on X as b runs through any bounded subset B of E. Hence $I(f)$ has the defining property of elements of $\mathcal{LE}(E, CV(X))$, but it remains to show that $I(f)$ is continuous from E into $CV(X)$ for $f \in CV(X, E'_b)$.

Again, fix $v \in V$. Since $(vf)(X)$ is bounded in E'_b and since E is quasibarrelled, $(vf)(X)$ is equicontinuous on E. So there exists a 0-neighborhood U in E with $(vf)(X) \subset U^\circ$. Hence $[I(f)](U)$ belongs to the unit ball of the seminorm p_v on $CV(X)$, and thus, finally, $I(f) \in \mathcal{LE}(E, CV(X))$.

Now fix $v \in V$ and B bounded in E, and take $f \in CV(X, E'_b)$. A glance at the equality

$$\begin{aligned} q_{v,p_B}(f) &= \sup_{x\in X} v(x) p_B(f(x)) = \sup_{x\in X}\sup_{b\in B} v(x)|[f(x)](b)| \\ &= \sup_{b\in B}\sup_{x\in X} v(x)|[I(f)](b)(x)| = \sup_{b\in B} p_v([I(f)](b)) = q_{B,v}(I(f)) \end{aligned}$$

reveals that I is a topological isomorphism of $CV(X, E'_b)$ into $\mathcal{L}_b(E, CV(X))$.

At this point, let us see that an arbitrary fixed $T \in \mathcal{LE}(E, CV(X))$ belongs to the image of I. It will suffice to show that the function f, defined by $x \to \delta_x \circ T$, is continuous from X into E'_b: For fixed $v \in V$ and B bounded in E, $T(B)$ is bounded in $CV(X)$, and hence $\sup_{b\in B} p_v(T(b)) < \infty$. But this implies that

$$\begin{aligned} \sup_{x\in X} q_B((vf)(x)) &= \sup_{x\in X} v(x) q_B(\delta_x \circ T) = \sup_{x\in X}\sup_{b\in B} v(x)|(\delta_x \circ T)(b)| \\ &= \sup_{b\in B}\sup_{x\in X} |(vT)(b)(x)| = \sup_{b\in B} p_v(T(b)) < \infty, \end{aligned}$$

whereby $(vf)(X)$ is bounded in E'_b. The desired continuity will then yield $f \in CV(X, E'_b)$, and it is clear that $I(f) = T$.

But note that, by definition of $\mathcal{LE}(E, CV(X))$, the image $T(B)$ of each bounded subset B of E is equicontinuous on X. Hence $x \to \delta_x \circ T$ is continuous from X into E'_b, as desired. □

Taking the discrete topology on an index set I and $V = \overline{V}$, Proposition 1 clearly implies [3], I, Proposition 2.13.(a). It also generalizes [7], Proposition A and, at the same time, characterizes the range of the topological isomorphism $CB(X, E'_b) \to \mathcal{L}(E, CB(X))$. Here comes the result for weighted spaces of vector valued holomorphic functions.

2. Theorem. *Let G denote an open subset of $\mathbb{C}^N$ and V a Nachbin family on G which satisfies $V \geq W$. If the locally convex space E is quasibarrelled, then we have the topological isomorphism*

$$HV(G, E'_b) = \mathcal{L}_b(E, HV(G)).$$

PROOF. The restriction of the canonical isomorphism I from Proposition 1 to $HV(G, E'_b)$ takes its values in $\mathcal{L}(E, HV(G))$ by the definition of (weak) holomorphy. Moreover, since $V \geq W$ means that the topology of $HV(G)$ is stronger than uniform convergence on compact sets, Montel's theorem and the Arzelà-Ascoli theorem combine to show that, in the notation of Proposition 1,

$$\mathcal{L}(E, HV(G)) = \mathcal{LE}(E, HV(G)).$$

At this point, it is enough to prove that, for arbitrary fixed $T \in \mathcal{L}(E, HV(G))$, $z \to \delta_z \circ T$ defines a holomorphic mapping on G with values in E'_b. But Große-Erdmann [12], Theorem

5.2 (which then also holds for open sets $G \subset \mathbb{C}^N$) shows that it suffices to prove $z \to e''(\delta_z \circ T)$ holomorphic for each e'' in a *separating* subset S of E''; i.e., $e' \in E'$, $e' \neq 0$ implies that there is $s \in S$ with $s(e') \neq 0$. But $S := E$ certainly is a separating subset of E'', and for arbitrary $e \in E$ the function $z \to (\delta_z \circ T)(e) = [T(e)](z)$ does belong to $HV(G)$.

Here is a different proof of the last point which, however, requires that E is in addition quasicomplete and hence barrelled: As we have seen that for arbitrary $e \in E$ the function $z \to (\delta_z \circ T)(e) = [T(e)](z)$ is holomorphic, the mapping $z \to \delta_z \circ T$ is already weakly holomorphic into E'_τ; i.e., E', endowed with the Mackey topology $\tau(E', E)$. If E is barrelled, E'_τ and E'_b have the same bounded sets so that the topologies $\tau(E', E)$ and $\beta(E', E)$ also yield the same holomorphic functions with values in E'. (This last point follows from Grothendieck's work; e.g., see [13], page 40 or [12], Theorem 3.1.) □

Theorem 2 obviously generalizes [7], Proposition B to weighted spaces. For the special case that G is a balanced domain, V consists only of continuous radial functions, $HV_0(G)$ contains the polynomials and that $HV(G)$ is bornological, Theorem 2 was proved in [6], Corollary 3.8 by transposing the vector valued duality $\mathcal{L}_b((HV_0(G))'_b, E) = HV(G, E)$ (for the proof of which the extra assumptions were needed).

In the spirit of [1], [2], [8] and [9] (where interesting examples can be found), we will now note a "semi-axiomatic generalization" of Theorem 2.

Let X be a completely regular k-space and $A(X)$ a linear subspace of $C(X)$ which is closed with respect to the compact-open topology τ_0 (hence τ_0-complete) and has the semi-Montel property; that is, each bounded subset of $(A(X), \tau_0)$ is relatively compact. For a quasibarrelled locally convex space E, let

$$A(X, E'_b) := \{f \in C(X, E'_b);\ x \to [f(x)](e) \text{ belongs to } A(X)\ \ \forall e \in E\}.$$

Fix a Nachbin family V on X which satisfies $V \geq W$ and put, with the induced weighted topology,

$$AV(X) := CV(X) \cap A(X),\ \ AV(X, E'_b) := CV(X, E'_b) \cap A(X, E'_b).$$

3. Corollary. *In the situation described above, we again obtain the canonical topological isomorphism*

$$AV(X, E'_b) = \mathcal{L}_b(E, AV(X)).$$

In fact, Corollary 3 follows along the lines of the proof of Theorem 2. (For the Ascoli theorem on k-spaces, see [16], 7.18.)

One is tempted to identify, in a similar way as in Proposition 1, $CV(X, E)$ with a space of linear operators from E'_b into $CV(X)$, even when E is not necessarily quasibarrelled. It will soon turn out that such an isomorphism is rather complicated, and hence less easily applicable, but that, by serendipity, one can get a better understanding of Proposition 1 in this way. In the sequel, the subscript e in $\mathcal{L}_e(E'_b, CV(X))$ stands for the topology of uniform convergence on the equicontinuous subsets of E'. This topology is given by the system $\{q_{p,v};\ p \in cs(E),\ v \in V\}$ of seminorms, where

$$q_{p,v}(T) := \sup_{e' \in B_p^\circ} p_v(T(e')) = \sup_{e' \in B_p^\circ} \sup_{x \in X} v(x)|[T(e')](x)|,\ T \in \mathcal{L}_e(E'_b, CV(X)).$$

4. Proposition. *For any locally convex space E, $CV(X,E)$ is topologically isomorphic to the following topological subspace of $\mathcal{L}_e(E'_b, CV(X))$:*

$$\mathcal{LE}^E(E'_b, CV(X)) := \{T \in \mathcal{L}(E'_b, CV(X));$$
$$T(H) \text{ equicontinuous on } X\ \forall H \text{ equicontinuous } \subset E' \text{ and } \delta_x{\circ}T \in E\ \forall x \in X\}.$$

The canonical isomorphism I of $CV(X,E)$ into $\mathcal{L}_e(E'_b, CV(X))$ is given by

$$[I(f)](e') : x \longrightarrow e'(f(x)),\ e' \in E',\ f \in CV(X,E).$$

PROOF. For $f \in CV(X,E)$, $I(f)$ clearly is a linear mapping from E' into $CV(X)$. To show that $I(f) : E'_b \to CV(X)$ is continuous, fix $v \in V$. It suffices to find a bounded subset B of E such that $[I(f)](B^\circ) \subset B_v$, and taking $B := (vf)(X)$ certainly does the job. Next, for any equicontinuous $H \subset E'$,

$$[I(f)](H) = \{x \to (e'{\circ}f)(x);\ e' \in H\}$$

is indeed equicontinuous on X, and since

$$\delta_x{\circ}[I(f)] : e' \to \delta_x(e'{\circ}f) = e'(f(x)),$$

we obtain, in the bidual $E'' = (E'_b)'_b$ of E, $\delta_x{\circ}[I(f)] = f(x) \in E$ for each $x \in X$, as desired. Finally, a glance at the following equality will reveal that I is a topological isomorphism of $CV(X,E)$ into $\mathcal{L}_e(E'_b, CV(X))$:

$$\begin{aligned} q_{p,v}(I(f)) &= \sup_{e' \in B_p^\circ} \sup_{x \in X} v(x)|[I(f)](e')(x)]| \\ &= \sup_{x \in X} v(x) \sup_{e' \in B_p^\circ} |(e'{\circ}f)(x)| \;=\; \sup_{x \in X} v(x)p(f(x)) \;=\; q_{v,p}(f). \end{aligned}$$

At this point, it remains to show that, given $T \in \mathcal{LE}^E(E'_b, CV(X))$, $x \to \delta_x{\circ}T$ defines an element $f \in CV(X,E)$ for which $I(f) = T$. Note that, by definition, f equals $x \to T'(\delta_x)$, where $T' : CV(X)' \to E''$ denotes the transposed of T. However, since T belongs to $\mathcal{LE}^E(E'_b, CV(X))$, f takes its values in the subspace E of E'' and is continuous from X into E: Fix $x_0 \in X$, $p \in cs(E)$ and $\varepsilon > 0$. We claim that there is a neighborhood $U = U(x_0)$ of x_0 such that $p(f(x) - f(x_0)) \leq \varepsilon$ for all $x \in U$. To see this, note that B_p° is an equicontinuous subset of E' whose image under T must be equicontinuous on X. That is, there exists a neighborhood $U = U(x_0)$ of x_0 such that $|[T(e')](x) - [T(e')](x_0)| \leq \varepsilon$ for $x \in U$ and $e' \in B_p^\circ$. It follows that for each $x \in U$,

$$\sup_{e' \in B_p^\circ} |e'(f(x) - f(x_0))| = \sup_{e' \in B_p^\circ} |[T(e')](x) - [T(e')](x_0)| \leq \varepsilon,$$

whence $f(x) - f(x_0) \in \varepsilon B_p^{\circ\circ}$, where the bipolar is taken in $< E, E' >$. But since $B_p^{\circ\circ} = B_p$ by the bipolar theorem, $p(f(x) - f(x_0)) \leq \varepsilon$ for each $x \in U$, and we have established our claim. Moreover, since, for any $v \in V$ and $p \in cs(E)$,

$$\begin{aligned} q_{v,p}(f) &= \sup_{x \in X} p(v(x)f(x)) = \sup_{x \in X} \sup_{e' \in B_p^\circ} |v(x)[T(e')](x)| \\ &= \sup_{e' \in B_p^\circ} p_v(T(e')) \;=\; q_{p,v}(T) < \infty, \end{aligned}$$

f indeed belongs to $CV(X,E)$. The proof is finished by remarking that $I(f)=T$ can be checked very easily. $\Box$

Of course, if E is quasibarrelled, $\mathcal{L}_e(E'_b, CV(X)) = \mathcal{L}_b(E'_b, CV(X))$. And a similar (even easier) argument as in the proof of Theorem 2 shows that, for an open set $G \subset \mathbb{C}^N$,

$$\mathcal{L}\mathcal{E}^E(E'_b, HV(G)) = \{T \in \mathcal{L}(E'_b, HV(G));\ \delta_z \circ T \in E\ \forall z \in G\},$$

whereby $I_{|HV(G,E)}$ is a topological isomorphism of $HV(G,E)$ onto this space.

In fact, since E is a topological subspace of E'' whenever E is quasibarrelled, part of Proposition 1 is a consequence of Proposition 4 as follows: First replace E by the strong dual E'_b of a quasibarrelled space E in Proposition 4 to obtain the topological isomorphism

$$I: CV(X, E'_b) \longrightarrow \mathcal{L}\mathcal{E}^{E'}((E'_b)'_b, CV(X)) \subset \mathcal{L}_e(E'', CV(X)).$$

Next, if each $I(f)$, $f \in CV(X, E'_b)$, is restricted to the topological subspace E of E'', the present isomorphism I yields the mapping I of Proposition 1. Since for each bounded set $B \subset E$ the polar B° is a 0-neighborhood in E'_b, hence $B \subset B^{\circ\bullet}$ ($^\bullet$ denoting the polar in $< E', E'' >$) equicontinuous in E'' and since, conversely, any subset B of E which is equicontinuous in E'' must certainly be bounded in E, one can use Proposition 4 to deduce one direction of Proposition 1. (To get a full proof in this way, one would still have to show that any $T \in \mathcal{L}\mathcal{E}(E, CV(X))$ extends to a mapping in $\mathcal{L}\mathcal{E}^{E'}(E'', CV(X))$.)

In principle, a topological isomorphism as in Theorem 2 (or Corollary 3) can be utilized to show that $HV(G, E'_b)$ (or $AV(X, E'_b)$) satisfies, or does not satisfy, certain locally convex properties. Note that in [7], Proposition C the isomorphism $H^\infty(G, E'_b) = \mathcal{L}_b(E, H^\infty(G))$ was already applied to prove that there is a reflexive Fréchet space E such that $H^\infty(D, E'_b)$ is not a (DF)-space, where $D =$ open unit disk in $\mathbb{C}$. On the other hand, Peris [18], Corollary 4.3.(1) has the consequence that $HV(G, E'_b)$ is a bornological (DF)-space if E is metrizable and quasinormable by operators (i.e., property (QNo) holds) and if $HV(G)$ is a (DF)-space with $(QNo)'$ (i.e., the strict Mackey condition by operators), which happens in particular whenever V consists of the multiples of one single continuous weight v – so that $Hv(G, E'_b)$ is indeed bornological (DF) whenever the metrizable space E has (QNo). For examples of spaces with (QNo) and $(QNo)'$ and for permanence properties of (QNo) and $(QNo)'$ with respect to spaces $\mathcal{L}_b(E,F)$ of operators, we refer to [18]. Note also that it directly follows from [11], Proposition 1.8 that $HV(G, E'_b)$ (resp., $AV(X, E'_b)$) is always quasicomplete under the conditions of Theorem 2 (resp., of Corollary 3). Here are some other examples (in which E is always assumed quasibarrelled): [15], 12.4.2 implies that $HV(G, E'_b)$ is a Fréchet space whenever E is a gDF-space and $HV(G)$ is a Fréchet space, and [15], 16.4.1 resp. 21.5.9.(b) (combined with 21.2.3) shows that $HV(G, E'_b)$ is Schwartz resp. nuclear whenever both E'_b and $HV(G)$ are. Finally, it is easily deduced from [15], 16.4.3 that $HV(G, E'_b)$ is a Fréchet-Montel space whenever both E'_b and $HV(G)$ are. (Note that this already is an ε-product result. Also see [5], 5.6.(2) for a condition (M') which implies that $HV(G)$ is semi-Montel and remark that (M') also implies $HV(G) = HV_0(G)$.)

Let us now restrict the attention to Nachbin families $V \geq W$ on an open set $G \subset \mathbb{C}^N$ which consist only of non-negative continuous functions and for which the space $HV(G)$ is bornological. Then by [6], 1.3 (also see [4], Section 3.A) $HV(G)$ is the strong dual of the complete barrelled space

$$P = (HV(G), \bar{\tau})'_b = (HV(G), \bar{\tau})'_c,$$

where $\bar{\tau}$ is the strongest locally convex topology coinciding with the compact-open topology on all bounded subsets of $HV(G)$. We remind that $\bar{\tau}$ is a semi-Montel topology, but not quasibarrelled in general. If $HV(G)$ is a Fréchet space (e.g., if V has the form $\{\lambda v_n;\ \lambda > 0,\ n \in I\!N\}$ for an increasing sequence of positive continuous functions v_n on G), then P is a complete barrelled (DF)-space, and [4], Corollary 5 shows how additional properties of $HV(G)$ are reflected in topological properties of the predual P. In particular, for quasinormable Fréchet spaces $HV(G)$, P must be a boundedly retractive (LB)-space (and hence bornological). In the case that V consists of all positive multiples of a single positive continuous function v, one can show that $(Hv(G), \bar{\tau}) = HW(G) = HW_0(G)$ holds topologically and is a complete semi-Montel (gDF)-space, where $W = B_0^+(G)V$ and $B_0^+(G)$ is the system of all non-negative continuous functions which vanish at infinity on G. Noting that transposition (and restriction) gives $\mathcal{L}_b(E, P'_b) = \mathcal{L}_b(P, E'_b)$ whenever E is quasibarrelled (see [11], 5.10), we end up with

5. Corollary. *If the Nachbin family $V \geq W$ on the open set $G \subset \mathbb{C}^N$ consists of non-negative continuous functions, if $HV(G)$ is bornological, and if E is a quasibarrelled space, then*

$$HV(G, E'_b) = \mathcal{L}_b(E, P'_b) = \mathcal{L}_b(P, E'_b) = \mathcal{L}_b((HV(G), \bar{\tau})'_b, E'_b).$$

The canonical isomorphism J of $\mathcal{L}_b(P, E'_b)$ onto $HV(G, E'_b)$ is given by

$$J(T) = T \circ \Delta : z \to T(\delta_z),\ \ T \in \mathcal{L}(P, E'_b),$$

where $\Delta : z \to \delta_z$ is considered as a mapping from G into $P = (HV(G), \bar{\tau})'$.

If, in addition to the conditions in Corollary 5, G is balanced, all the weights $v \in V$ are radial (in the sense of [6]), and if $HV_0(G)$ contains the polynomials, then [6], Theorem 1.5.(d) yields $HV(G, E'_b) = \mathcal{L}_b((HV_0(G))'_b, E'_b)$. However, remember that in [6], Proposition 3.7 it was already proved more generally that, for any quasicomplete locally convex space E, the vector valued duality $\mathcal{L}_b((HV_0(G))'_b, E) = HV(G, E)$ holds under the above conditions on G and V.

3. Operator representations for function spaces on product sets

In this last section we prove operator representations for weighted spaces of continuous and holomorphic functions in several variables and again start with the case of continuous functions.

6. Proposition. *Let X_1, X_2 denote completely regular $k_{I\!R}$-spaces such that the product $X_1 \times X_2$ is again a $k_{I\!R}$-space. (This holds, in particular, if both X_1 and X_2 are metrizable – or hemicompact; i.e., with a countable basis of compact sets –, or if one of the spaces is* *locally compact.) Let V_i be a Nachbin family on X_i with $V_i \geq W$ for $i = 1, 2$. Then we have the canonical topological isomorphism*

$$I : f \longrightarrow ((x_1, x_2) \to [f(x_1)](x_2))$$

of $CV_1(X_1, CV_2(X_2))$ into $C(V_1 \otimes V_2)(X_1 \times X_2)$. In general, this isomorphism is not surjective. The range of I is the subspace $C(V_1 \otimes V_2)^{1e}(X_1 \times X_2)$, defined by

$$\{g \in C(V_1 \otimes V_2)(X_1 \times X_2);\ \{v_2(x_2)g(., x_2);\ x_2 \in X_2\} \text{ is equicontinuous on } X_1\ \forall v_2 \in V_2\},$$

where the assumption 'equicontinuous on X_1' can be replaced by 'equicontinuous on each compact subset of X_1' whenever X_1 is a k-space.

Proof. Because of $V_2 \geq W$, any function $f \in C(X_1, CV_2(X_2))$ a fortiori belongs to $C(X_1, (C(X_2), \tau_0))$. Since $X_1 \times X_2$ is a $k_{\mathbb{R}}$-space, we have the well-known isomorphism for the compact-open topologies $C(X_1, C(X_2)) = C(X_1 \times X_2)$ (cf. [1], p. 48, Corollary 35), which implies $I(f) \in C(X_1 \times X_2)$. Now, for $f \in CV_1(X_1, CV_2(X_2))$, $v_1 \in V_1$ and $v_2 \in V_2$,

$$\begin{aligned} p_{v_1 \otimes v_2}(I(f)) &= \sup_{(x_1,x_2) \in X_1 \times X_2} v_1(x_1)v_2(x_2)|[f(x_1)](x_2)| \\ &= \sup_{x_1 \in X_1} v_1(x_1) \sup_{x_2 \in X_2} v_2(x_2)|[f(x_1)](x_2)| \ = \ q_{v_1, p_{v_2}}(f) \end{aligned}$$

shows that $I(f)$ belongs to $C(V_1 \otimes V_2)(X_1 \times X_2)$ and that I is a topological isomorphism of $CV_1(X_1, CV_2(X_2))$ into $C(V_1 \otimes V_2)(X_1 \times X_2)$. Grothendieck [14], p. 18 already pointed out that, for non-compact X_1 and X_2, I is in general not surjective from $CB(X_1, CB(X_2))$ to $CB(X_1 \times X_2)$, where all the spaces of continuous bounded functions are equipped with the topology of uniform convergence. Clearly, if $f \in C(X_1, CV_2(X_2))$, then for each $v_2 \in V_2$ the set $\{v_2(x_2)[f(.)](x_2);\ x_2 \in X_2\}$ is equicontinuous on X_1. The inverse mapping $J = I^{-1}$: $g \longrightarrow (x_1 \to (x_2 \to g(x_1, x_2)))$ of I can be defined on the whole space $C(V_1 \otimes V_2)(X_1 \times X_2)$, and it remains to see that $J(g)$ belongs to $CV_1(X_1, CV_2(X_2))$ for each $g \in C(V_1 \otimes V_2)(X_1 \times X_2)$ with the additional equicontinuity assumption. The above equality of seminorms shows that it suffices to prove $J(g) \in C(X_1, CV_2(X_2))$, and this is clear if g belongs to the subspace $C(V_1 \otimes V_2)^{1e}(X_1 \times X_2)$. – The last part is obvious because equicontinuity on a k-space is equivalent to equicontinuity on all compact subsets of this space. □

Here is the main result of Section 3: The topological isomorphism from Proposition 6 is much more natural for spaces of holomorphic functions.

7. Theorem. *Let $G_1 \subset \mathbb{C}^N$, $G_2 \subset \mathbb{C}^M$ be open, and let V_i be a Nachbin family on G_i with $V_i \geq W$ for $i = 1, 2$. In this case, there is a (surjective) topological isomorphism*

$$HV_1(G_1, HV_2(G_2)) = H(V_1 \otimes V_2)(G_1 \times G_2).$$

Proof. First note that if $g \in H(V_1 \otimes V_2)(G_1 \times G_2)$, then the set $\{v_2(z_2)g(., z_2);\ z_2 \in G_2\}$ is bounded in $HV_1(G_1)$ for each $v_2 \in V_2$. Thus, $V_1 \geq W$ implies, via Montel's theorem and the Arzelà-Ascoli theorem, that for any $g \in H(V_1 \otimes V_2)(G_1 \times G_2)$ the additional equicontinuity property of the functions in the range of the isomorphism I from Proposition 6 is always satisfied. At this point, in view of Hartogs' theorem that holomorphy in several variables is equivalent to separate holomorphy (which easily implies that I maps $H(G_1, HV_2(G_2))$ into $H(G_1 \times G_2)$) and in view of Proposition 6, it is clearly enough to show that for any $g \in H(V_1 \otimes V_2)(G_1 \times G_2)$ the function $J(g) \in C(G_1, HV_2(G_2))$ is (weakly) holomorphic from G_1 into $HV_2(G_2)$. But [12], Theorem 5.2 now implies that it suffices to show $\mu \circ J(g)$ holomorphic for each μ from a separating set $S \subset (HV_2(G_2))'$. $S := \{\delta_{z_2};\ z_2 \in G_2\}$ is such a separating subset, and of course $\delta_{z_2} \circ g = g(., z_2)$ is holomorphic for each $z_2 \in G_2$. $\square$

The isomorphism $H^\infty(D, H^\infty(D^{n-1})) = H^\infty(D^n)$, $n \in I\!N$, which is a simple special case of Theorem 7, had been proved (in a different way) by Cutrer [10]. – We close the article with the following corollary of Theorem 7.

8. Corollary. *Let $G_1 \subset \mathbb{C}^N$, $G_2 \subset \mathbb{C}^M$ be open, let V_i be a Nachbin family on G_i with $V_i \geq W$ for $i = 1, 2$, and assume that V_2 consists of non-negative continuous functions and that $HV_2(G_2)$ is bornological. Let P_2 denote the predual of $HV_2(G_2)$. Then we have the following topological isomorphisms*

$$H(V_1 \otimes V_2)(G_1 \times G_2) = HV_1(G_1, (P_2)'_b) = \mathcal{L}_b(P_2, HV_1(G_1)).$$

If in addition G_2 is balanced, all the weights $v_2 \in V_2$ are radial, and if $H(V_2)_0(G_2)$ contains the polynomials, then the spaces above are also topologically isomorphic to

$$\mathcal{L}_b((H(V_2)_0(G_2))'_b, HV_1(G_1)).$$

The results in this article are part of the second-named author's dissertation.

References

[1] K.-D. Bierstedt, The approximation property for weighted function spaces; Tensor products of weighted spaces, pp. 3–25, 26–58 in *Function Spaces and Dense Approximation* (Proc. Conference Bonn 1974), *Bonner Math. Schriften* **81** (1975).

[2] K.-D. Bierstedt, The approximation-theoretic localization of Schwartz's approximation property for weighted locally convex function spaces and some examples, pp. 93–149 in *Functional Analysis, Holomorphy, and Approximation Theory* (Proc. Conference Rio de Janeiro 1978), Springer *Lecture Notes in Math.* **843** (1981).

[3] K.D. Bierstedt, J. Bonet, Dual density conditions in (DF)-spaces, I., *Results Math.* **14** (1988), 241–274; II., *Bull. Soc. Roy. Sci. Liège* **57** (1988), 567–589.

[4] K.D. Bierstedt, J. Bonet, Biduality in Fréchet and (LB)-spaces, pp. 113–133 in *Progress in Functional Analysis* (Proc. Conference Peñíscola 1990), *North-Holland Mathematics Studies* **170** (1992).

[5] K.D. Bierstedt, R. Meise, Distinguished echelon spaces and the projective description of weighted inductive limits of type $\mathcal{V}_d\mathcal{C}(X)$, pp. 169–226 in *Aspects of Mathematics and its Applications* (dedicated to L. Nachbin), *North-Holland Mathematical Library* **34** (1986).

[6] K.D. Bierstedt, J. Bonet, A. Galbis, Weighted spaces of holomorphic functions on balanced domains, *Michigan Math. J.* **40** (1993), 271–297.

[7] K.D. Bierstedt, J. Bonet, J. Schmets, (DF)-spaces of type $CB(X,E)$ and $C\overline{V}(X,E)$, *Note Mat.* **10**, Suppl. n. 1 (dedicated to the memory of G. Köthe) (1990), 127–148.

[8] K.-D. Bierstedt, B. Gramsch, R. Meise, Approximationseigenschaft, Lifting und Kohomologie bei lokalkonvexen Produktgarben, *Manuscripta Math.* **19** (1976), 319–364.

[9] K.D. Bierstedt, R. Meise, W.H. Summers, A projective description of weighted inductive limits, *Trans. Amer. Math. Soc.* **272** (1982), 107–160.

[10] W. Cutrer, Vitushkin's localization operator and holomorphic functions on polydiscs, *Duke Math. J.* **39** (1972), 737–744.

[11] S. Dierolf, On spaces of continuous linear mappings between locally convex spaces, *Habilitationsschrift* (Univ. München 1983), *Note Mat.* **5** (1985), 147–255.

[12] K.-G. Große-Erdmann, The Borel-Okada theorem revisited, *Habilitationsschrift* (Fern-Univ. Hagen 1992).

[13] A. Grothendieck, Sur certains espaces de fonctions holomorphes I, *J. Reine Angew. Math.* **192** (1953), 35–64.

[14] A. Grothendieck, *Espaces vectoriels topologiques*, São Paulo (1954).

[15] H. Jarchow, *Locally Convex Spaces*, Mathematische Leitfäden, B.G. Teubner (1981).

[16] J.L. Kelley, *General Topology*, van Nostrand (1955, reprint 1969).

[17] L. Nachbin, *Elements of Approximation Theory*, van Nostrand Mathematical Studies **14** (1967).

[18] A. Peris, Quasinormable spaces and the problem of topologies of Grothendieck, *Ann. Acad. Sci. Fenn., Ser. A.I. Math.* **19** (1994), 167–203.

Authors' address:
FB 17, Mathematik
Univ. Paderborn
D-33095 Paderborn, Germany

e-mail addresses:
klausd@uni-paderborn.de, toelva@uni-paderborn.de