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Squared Eigenfunctions of the (Modified) KP Hierarchy and Scattering Problems of Loewner Type

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Abstract

It is shown that products of eigenfunctions and (integrated) adjoint eigenfunctions associated with the (modified) Kadomtsev-Petviashvili (KP) hierarchy form generators of a symmetry transformation. Linear integro-differential representations for these symmetries are found. For special cases the corresponding nonlinear equations are the compatibility conditions of linear scattering problems of Loewner type. The examples include the 2+1-dimensional sine-Gordon equation with space variables occuring on an equal footing introduced recently by Konopelchenko and Rogers. This equation represents a special squared eigenfunction symmetry of the Ishimori hierarchy.

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1 Introduction

The solvable nonlinear equations of soliton theory are given as the compatibility conditions of linear differential equations (see e.g. the recent books [1] or [14] for a state of the art presentation). It was observed at the early stages of this theory that "squared eigenfunctions", i.e. products of solutions of the linear problems, play a role in the treatment of such systems. Already in his fundamental paper [24] Lax observed for the Korteweg-de Vries (KdV) equation that squares of eigenfunctions of the Schrödinger equation may be regarded as spectral gradients, i.e. they have the properties of gradients of conservation laws. Through the Hamiltonian formalism a corresponding symmetry transformation is generated. Analogous results for soliton equations in 2+1 dimensions are known (e.g. [26, 34]).

A systematic "nonlinearization" procedure was proposed by Cao [4], in which the potentials of the scattering problem are identified with squared eigenfunctions. Upon elimination of the potentials the linear problems result in nonlinear equations for the eigenfunctions, which provide new examples of solvable hierarchies. It turns out that the constraint between potentials and eigenfunctions may be understood as a symmetry reduction, in which the symmetry generated by squared eigenfunctions is identified with a flow of the soliton hierarchy.

Recently there has been much interest in this kind of "nonlinearization", which may be used to reduce solvable hierarchies in 2+1 dimensions to interesting systems in 1+1 dimensions [5, 6, 10, 20, 22, 26, 39, 40, 42, 45]. In particular, in the context of Sato's construction of the KP hierarchy [9, 13, 32, 35, 36], the constraint admits a simple interpretation in terms of the pseudo-differential Lax operator [5, 31, 33, 41, 43]. A systematic interpretation of the reduction as a symmetry constraint was found.

So far, the symmetry generated by squared eigenfunctions was mainly used to characterize admissible constraints of the nonlinear equations. It was pointed out by Orlov [33] that these symmetries are interesting equations in their own right.

Indeed, following Loewner's work on infinitesimal Bäcklund transformations, Konopelchenko and Rogers recently considered the nonlinear systems given by the compatibility conditions of hyperbolic second order scattering problems [18, 19]. A variety of both known as well as novel systems were found, the most remarkable of which is an integrable 2+1-dimensional version of the sine/sinh-Gordon equation symmetric in the two space variables. These systems were shown to be treatable by the $\bar{\partial}$ formalism of the Inverse Scattering Transform Method [15, 19], Bäcklund transformations [21] and localized solutions were found [11, 37].

We will show that these equations are nothing but special symmetries of the multicomponent (modified) KP hierarchy, which are generated by squared eigenfunctions. More generally, a systematic formulation of this infinitesimal symmetry is proposed, which can be applied to any linear problem associated with an arbitrary matrix differential operator.

In the context of soliton theory in 2+1 dimensions, the most interesting linear equations are those giving rise to the multicomponent KP hierarchy and its modified version. In this case the eigenfunctions generate an infinitesimal symmetry of the dressing operator, from which the nonlinear soliton equations are derived. In the modified case the underlying linear operators have a special form in that no zero-order terms are present. The general symmetry has to be modified in order to preserve this special structure.

In Section 2 we briefly review the Loewner system and discuss the two basic cases which later are shown to be associated with the KP hierarchy and the modified KP hierarchy respectively. In Section 3 the dressing formalism is reviewed, which allows to formulate the multicomponent (modified) KP hierarchy in a very simple and compact way. It represents an algebraic version of the Zakharov-Shabat dressing technique [44] and is in close analogy to Sato's construction of the scalar KP. It is briefly demonstrated how the (modified) KP equation, the Davey-Stewartson equation and the Ishimori equation reside in this setup. In Section 4 the general formulation of the squared eigenfunction symmetry is given. Pseudo-differential symbols provide a simple and convenient tool to describe this invariance. In Section 5 it is indicated that a Lax formulation exists for the symmetry. In contrast to the other equations of the hierarchy this linear representation is not of a purely differential nature but is given by an integro-differential equation. It is shown that the general infinitesimal generator of the squared eigenfunction symmetry can be specialized to a simple set of *ordinary* differential equations, which can be solved explicitly. In Section 6 the Loewner triad is derived from a special case of the symmetry. This establishes the connection between the Loewner system and the (modified) KP hierarchy. In Section 7 certain reductions are discussed which are given as constraints between the linear problems and their adjoints. It is shown that such constraints can be preserved by the symmetry. Finally, in Section 8, various examples are worked out. In particular, the scalar Zakharov-Manakov triad ("Darboux system") is shown to be a squared eigenfunction symmetry of the Davey-Stewartson hierarchy, the 2+1-dimensional sine-Gordon system is shown to be connected with the Ishimori hierarchy and (via constraints) with the modified Nizhnik-Veselov-Novikov hierarchy.

2 The Generalized Loewner system

In 1952, Loewner introduced a linear triad (Loewner system) in connection with infinitesimal Bäcklund transformations in a gasdynamics context [25]. Here, we shall be concerned with its generalization to the "LKR system" as recently established by Konopelchenko and Rogers [18]. One purpose of this paper is to reformulate the (generalized) Loewner system in terms of a squared eigenfunction symmetry of the well-known multicomponent (modified) KP hierarchy. Before we do so we briefly review the structure and specializations of the Loewner system as given in Refs. [18, 19].

In the following we shall be dealing with the triad system

$$\begin{aligned}\Phi_y &= S\Phi_x + P\Phi \ , \\ \Phi_{zx} &= U\Phi_z + V\Phi_x + W\Phi \ , \\ \Phi_{zy} &= \tilde{U}\Phi_z + \tilde{V}\Phi_y + \tilde{W}\Phi \ ,\end{aligned} \tag{2.1}$$

where $S, P, U, V, W, \tilde{U}, \tilde{V}, \tilde{W}$ and Φ are matrices and a vector respectively, dependent on the variables

x, y and z. Their dimension is as yet unspecified. It is readily shown that the compatibility conditions of the linear system (2.1) yield the nonlinear equations

$$
\begin{gathered}
S_z = [V, S] \quad , \\
P_z = \tilde{W} - SW + VP \quad , \\
U_y - \tilde{U}_x + [U, \tilde{U}] = 0 \quad , \\
V_y - \tilde{U}V - \tilde{W} + (UV + W - V_x)S = 0 \quad , \\
W_y - \tilde{W}_x + U\tilde{W} - \tilde{U}W + (UV + W - V_y)P = 0 \quad , \\
\tilde{U} = SU + P \quad , \quad \tilde{V} = V \quad .
\end{gathered}
\tag{2.2}
$$

In order to reveal a link of the above system to the KP and the modified KP hierarchy we have to consider two basic reductions of (2.2):

Firstly, the specialization $S = const$, $V = 0$ suggests the parametrization

$$
U = \phi_{1x}\phi_1^{-1} \quad , \quad \tilde{U} = \phi_{1y}\phi_1^{-1} \quad , \quad W = \phi_1\psi_1^\dagger \quad , \tag{2.3}
$$

so that the equations (2.2) reduce to the so-called $2+1$-dimensional non-Abelian sine-Gordon system

$$
P_z = [\phi_1\psi_1^\dagger, S] \quad , \quad \phi_{1y} = S\phi_{1x} + P\phi_1 \quad , \quad \psi_{1y} = S^\dagger\psi_{1x} - P^\dagger\psi_1 \tag{2.4}
$$

with matrix functions ϕ_1 and ψ_1. Here $\dagger$ denotes matrix transposition. In this representation the Loewner triad (2.1) takes the form

$$
\begin{aligned}
\Phi_y &= S\Phi_x + P\Phi \quad , \\
\Phi_{zx} &= \phi_{1x}\phi_1^{-1}\,\Phi_z + \phi_1\psi_1^\dagger\,\Phi \quad , \\
\Phi_{zy} &= \phi_{1y}\phi_1^{-1}\,\Phi_z + \phi_1\psi_1^\dagger\,S\Phi \quad ,
\end{aligned}
\tag{2.5}
$$

which gives rise to (2.4) via compatibility.

Secondly, the case $P = 0$ is most conveniently described by the parametrization

$$
\begin{aligned}
&U = \phi_{1x}\phi_1^{-1} \quad , \quad \tilde{U} = \phi_{1y}\phi_1^{-1} \quad , \quad V = \phi_1\omega_1^\dagger + \phi_2 \quad , \\
&W = \phi_{2x} - \phi_{1x}\phi_1^{-1}\phi_2 \quad , \quad \tilde{W} = \phi_{2y} - \phi_{1y}\phi_1^{-1}\phi_2 \quad ,
\end{aligned}
\tag{2.6}
$$

which produces the nonlinear system

$$
S_z = [\phi_1\omega_1^\dagger + \phi_2, S] \quad , \quad \phi_{1y} = S\phi_{1x} \quad , \quad \phi_{2y} = S\phi_{2x} \quad , \quad \omega_{1y} = S^\dagger\omega_{1x} \quad . \tag{2.7}
$$

On substitution of the representation (2.6) into the triad (2.1) we finally obtain

$$
\begin{aligned}
\Phi_y &= S\Phi_x \quad , \\
\Phi_{zx} &= \phi_{1x}\phi_1^{-1}\,\Phi_z + (\phi_1\omega_1^\dagger + \phi_2)\,\Phi_x + (\phi_{2x} - \phi_{1x}\phi_1^{-1}\phi_2)\,\Phi \quad , \\
\Phi_{zy} &= \phi_{1y}\phi_1^{-1}\,\Phi_z + (\phi_1\omega_1^\dagger + \phi_2)\,\Phi_y + (\phi_{2y} - \phi_{1y}\phi_1^{-1}\phi_2)\,\Phi \quad ,
\end{aligned}
\tag{2.8}
$$

which gives rise to (2.7) via compatibility. It has been shown in Refs. [18, 19] that specializations and/or $1+1$-dimensional reductions of the nonlinear equations (2.7) lead to integrable systems such as the principal chiral field model equation, the Toda lattice, the self-induced transparency equations and a 2+1-dimensional sine-Gordon system wherein the variables x and y occur on an equal footing.

In the following we will identify the hierarchies of "higher flows" (symmetries) of the LKR systems (2.4) and (2.7). The result is that these symmetries are nothing but the usual multicomponent KP hierarchy in the case of (2.4) and the multicomponent modified KP hierarchy in the case of (2.7). In fact, these LKR systems are just special cases of the more general integrable systems

$$
P_z = [\sum_{n=1}^{N} \phi_n\psi_n^\dagger, S] \quad , \quad \phi_{ny} = S\phi_{nx} + P\phi_n \quad , \quad \psi_{ny} = S^\dagger\psi_{nx} - P^\dagger\psi_n \quad , \tag{2.9}
$$

and

$$
S_z = [\sum_{n=1}^{N} \phi_n\omega_n^\dagger, S] \quad , \quad \phi_{ny} = S\phi_{1x} \quad , \quad \omega_{ny} = S^\dagger\omega_{nx} \quad , \tag{2.10}
$$

which we will identify as squared eigenfunction symmetries of the (modified) KP hierarchy. Here, an arbitrary number N of pairs of (adjoint) eigenfunctions ϕ_n, ψ_n (or ϕ_n, ω_n) can be used to generate these flows. We will identify Lax representations of (2.9) and (2.10) in Section 5, which are given by integro-differential equations. The special case of (2.9) with only one pair ϕ_1, ψ_1 is distinguished by the fact that the integro-differential linear problem can be reduced to the purely differential scattering problem (2.5). Similarly, with two pairs ϕ_1, ω_1 and $\phi_2, \omega_2 \equiv 1$ the Loewner triad (2.8) is obtained in the modified case as a specialization of the more general linear problem associated with an arbitrary number N of generating pairs.

In the next section we review the construction of the (modified) KP hierarchy, in which the isospectral flows associated with the linear problems $\Phi_y = S\Phi_x + P\Phi$ and $\Phi_y = S\Phi_x$ are found and represented in a compact way. This notation allows to introduce the concept of squared eigenfunction symmetries in a general and simple way, which will finally establish the connection of the (modified) KP hierarchy and the Loewner system.

3 The Multicomponent (Modified) KP Hierarchy

In this section we review the construction of a variety of integrable hierarchies in 2+1 dimensions. The method is an algebraic version of the Zakharov-Shabat dressing technique and essentially coincides with Sato's construction of the (scalar) KP hierarchy. The results provide a hierarchy of differential operators which encode various nonlinear soliton systems via zero-curvature equations. These operators are parametrized in terms of a dressing operator. This notation allows to encode the soliton hierarchies in a very compact way as a hierarchy of simple evolution equations for the dressing operator, for which the symmetry generated by squared eigenfunctions assumes a particularly simple form.

We consider pseudo-differential symbols over some finite-dimensional (matrix) algebra g_0 defined by the general Leibniz rule

$$\partial^n f \;=\; \sum_{i=0}^{\infty} \frac{n(n-1)...(n-i+1)}{1\times 2\times ...\times i}\,\frac{d^i f}{dx^i}\,\partial^{n-i} \quad , \tag{3.1}$$

where $f(x) \in g_0$ is a matrix function of the variable x. For positive n this is the usual Leibniz rule for differential operators, whereas for negative n an infinite series representing a formal asymptotic expansion is obtained. In particular, one has the "multiplication"

$$\partial^{-1} f \;=\; f\,\partial^{-1} - f_x\,\partial^{-2} + f_{xx}\,\partial^{-2} \pm ... \quad . \tag{3.2}$$

Together with $\partial\partial^{-1} = \partial^{-1}\partial = \textit{identity}$ this yields a well-defined associative multiplication on the space of pseudo-differential symbols $g = \{ \sum_{-\infty < i \ll \infty} u_i \partial^i \;\;,\;\; u_i \in g_0 \}$. By

$$\Big[\sum_i u_i \partial^i\Big]_{\geq k} \;=\; \sum_{i\geq k} u_i\partial^i \quad , \quad \Big[\sum_i u_i\partial^i\Big]_{<k} \;=\; \sum_{i<k} u_i\partial^i \tag{3.3}$$

an operator in g is projected onto differential orders larger and equal or smaller than a given order k. By

$$\Big[\sum_i u_i\partial^i\Big]_0 \;=\; u_0 \quad , \quad res(\sum_i u_i\partial^i) \;=\; u_{-1} \tag{3.4}$$

we denote the zero-order term and the usual residue of an operator.

<u>Warning:</u> We point out that in the following we will consider both <u>compositions</u> $M\Phi$ of operators $M \in g$ and functions Φ as well as the application of a differential operator $M \in [g]_{\geq 0}$ to a function. These two cases may easily be confused, so that we will use the notation $[M\Phi]_0$ when a differential operator acts on the function. This is consistent with the above notation, as the zero-order term of a composition $M\Phi$ is indeed nothing but the action of M on Φ.

The projections induce a decomposition $g = [g]_{\geq k} \oplus [g]_{<k}$ of the algebra into subspaces of symbols with differential orders larger and equal/smaller than k. We note that the two cases $k = 0$ and $k = 1$ are distinguished by the fact that both spaces $[g]_{\geq k}$ as well as $[g]_{<k}$ simultaneously form subalgebras. This is crucial for the commutativity of the flows constructed in the following, so that we will confine ourselves to the cases $k = 0$ and $k = 1$ for the rest of the paper. For the scalar case (i.e. $g_0 = \mathbb{R}$) also $k = 2$ yields a Lie algebra decomposition which gives rise to a further integrable hierarchy of Dym-type equations [17, 30].

We briefly review the construction of the multicomponent KP hierarchy and its modified version, which is in close analogy to Sato's construction of the KP hierarchy [9, 13, 32, 35, 36]. For the KP hierarchy zero-order terms are regarded to be of " positive" order, whereas in the modified case zero-order terms are considered to be part of the "negative" algebra.

The basic objects to construct the integrable equations of the (modified) KP hierarchy are the following dressing operators

$$\begin{aligned} (k=0): \quad & W = 1 + w_1\partial^{-1} + w_2\partial^{-2} + \ldots , \\ (k=1): \quad & W = w_0 + w_1\partial^{-1} + w_2\partial^{-2} + \ldots , \end{aligned} \tag{3.5}$$

where the matrix fields $w_0, w_1, w_2, \ldots \in g_0$ will be the dynamical fields of the (modified) KP hierarchy satisfying a certain set of nonlinear differential equations. The label $k=0$ is attributed to the KP hierarchy, whereas $k=1$ yields the modified KP hierarchy.

For an invertible leading order coefficient these operators can be inverted by means of an ansatz $W^{-1} = 1 + v_1\partial^{-1} + v_2\partial^{-2} + \ldots$ and $W^{-1} = w_0^{-1} + v_1\partial^{-1} + v_2\partial^{-2} + \ldots$ respectively. The requirement $WW^{-1} = 1$ leads to a straightforward recursive scheme for the coefficients v_j, from which they can be expressed as differential expressions of the coefficients w_j. In particular, the first terms of the expansion of W^{-1} are given as

$$\begin{aligned} (k=0): \quad & W^{-1} = 1 - w_1\partial^{-1} + (w_1^2 - w_2)\partial^{-2} + \ldots , \\ (k=1): \quad & W^{-1} = w_0^{-1} - w_0^{-1}w_1w_0^{-1}\partial^{-1} + (\ldots)\partial^{-2} + \ldots \quad . \end{aligned} \tag{3.6}$$

The coefficients w_j of W are to be endowed with a certain dynamics. For this we choose a collection of commuting elements in g_0, use them as coefficients of powers of the differential operator and define a set of commuting "undressed" (or "bare") operators

$$\mathcal{A} = \{ \ a_n\partial^n \ , \ n \in \mathbb{N} \ , \ a_n \in g_0 \ , \ [a_n, a_m] = 0 \ \} \quad . \tag{3.7}$$

With each element $A \in \mathcal{A}$ we associate a dynamics of the dressing operator W by requiring

$$W_{t_A} = -[WAW^{-1}]_{<k}W = -WA + [WAW^{-1}]_{\geq k}W \quad , \tag{3.8}$$

where t_A is to denote the evolution parameter ("time") of this flow. We note that this does indeed define a consistent dynamics for all the coefficients w_j in W, as the right-hand side of (3.8) is a pseudo-differential operator with highest differential order $k-1$. For $k=0$ the set of nonlinear equations for the coefficients w_j in W obtained from (3.8) is called the *multicomponent KP hierarchy* (associated with the collection $\mathcal{A}$ of undressed operators), whereas we will refer to the case $k=1$ as the *modified KP hierarchy*. The justification for these names is that fact that for the scalar case $g_0 = \mathbb{R}$ and $k=0$ the field w_1 satisfies the potential KP equation, whereas for $k=1$ the field $u = \ln(w_0)$ satisfies the potential version of the modified KP equation. These examples and other realizations of the nonlinear equations (3.8) are discussed in the following subsections.

It will be convenient to rephrase the system (3.8) as a hierarchy of zero-curvature formulations. One defines the differential operators

$$M(A) = [WAW^{-1}]_{\geq k} \tag{3.9}$$

as the dressed versions of the bare operators $A \in \mathcal{A}$. A straightforward computation yields the identities

$$\begin{aligned} W_{t_A t_B} - W_{t_B t_A} = -[\,W\,(A_{t_B} - B_{t_A} - [B,A])\,W^{-1}]_{<k} W \quad , \\ M(A)_{t_B} - M(B)_{t_A} - [M(B), M(A)] = [\,W\,(A_{t_B} - B_{t_A} - [B,A])\,W^{-1}]_{\geq k} \quad , \end{aligned} \tag{3.10}$$

for $k = 0, 1$ and any undressed operators A, B, which in principle may depend on the time variables. For these identities the Lie algebraic properties of the decomposition $g = [g]_{\geq k} \oplus [g]_{<k}$ are needed which distinguishes the cases $k = 0$ and $k = 1$. As the bare operators in $\mathcal{A}$ are assumed to be constant and commuting the flows (3.8) associated with two different undressed operators $A, B \in \mathcal{A}$ commute: $W_{t_A t_B} = W_{t_B t_A}$. The differential operators (3.9) satisfy the zero-curvature equation

$$M(A)_{t_B} - M(B)_{t_A} = [M(B), M(A)] \tag{3.11}$$

for any choice of the pair A, B.

We remark that also Lax formulations can be extracted from the dressing equations. For this one fixes an arbitrary element $C \in \mathcal{A}$ and defines a pseudo-differential Lax operator

$$L(C) = WCW^{-1} \quad . \tag{3.12}$$

With the identity

$$\begin{aligned} L(C)_{t_A} &= -[WAW^{-1}]_{<k} WCW^{-1} + WCW^{-1}[WAW^{-1}]_{<k} + WC_{t_A}W^{-1} \\ &= [M(A) - WAW^{-1}, L(C)] + WC_{t_A}W^{-1} = [M(A), L(C)] + W(C_{t_A} - [A,C])W^{-1} \end{aligned} \tag{3.13}$$

one finds that $L(C)$ satisfies the hierarchy of Lax equations $L(C)_{t_A} = [M(A), L(C)]$.

3.1 The KP hierarchy: Examples

3.1.1 The scalar case

The commuting hierarchy of equations (3.8, k=0) encapsulates the (scalar) KP equation and its higher flows in the simplest case where the matrix algebra g_0 for the coefficients of the dressing operator is chosen to be the set of scalar functions. We briefly review the results of Ref. [16], where the KP equation as well as other systems were derived from (3.8).

With the choices $A = \partial^2, B = \partial^3$ and the more convenient notation $y = t_A$, $t = t_B$, which will be used throughout this paper, the evolution equations (3.8) imply

$$\begin{aligned} w_{1y} &= w_{1xx} + 2w_{2x} - 2w_{1x}w_1 \quad , \\ w_{2y} &= w_{2xx} + 2w_{3x} - 2w_{1x}w_2 \quad , \end{aligned} \tag{3.14}$$

and

$$w_{1t} = w_{1xxx} + 3w_{2xx} + 3w_{3x} - 3(w_{1xx} + w_{2x} - w_1 w_{1x})w_1 - 3w_{1x}^2 - 3w_{1x}w_2 \quad . \tag{3.15}$$

The set of equations (3.14) can be used to express the fields w_2, w_3 in terms of w_1, its x- and y-derivatives. Insertion into (3.15) yields an equation for w_1 only, viz the (potential form of the) KP equation

$$4w_{1xt} = w_{1xxxx} - 12w_{1x}w_{1xx} + 3w_{1yy} \quad . \tag{3.16}$$

The associated scattering operators are calculated as

$$\begin{aligned} M(A) &= [W\partial^2 W^{-1}]_{\geq 0} &= \partial^2 - 2w_{1x} \ , \\ M(B) &= [W\partial^3 W^{-1}]_{\geq 0} &= \partial^3 - 3w_{1x}\partial - \tfrac{3}{2}(w_{1xx} + w_{1y}) \end{aligned} \tag{3.17}$$

which directly lead to (3.16) via the zero-curvature equation $M(A)_t - M(B)_y = [M(B), M(A)]$. For the higher flows of (3.16) one still considers the flow $W_y = -[W\partial^2 W^{-1}]_{<0}W$, from which all "auxiliary" fields $w_2, w_3, \ldots$ in W are expressed via w_1 and its x- and y-derivatives. Insertion into the higher equations $W_{t_n} = -[W\partial^n W^{-1}]_{<0}W,\ n \in \mathbb{N}$, yields the commuting evolution equations for w_1 which represent the hierarchy of higher KP flows.

3.1.2 The 2×2 case

We now proceed to the simplest multi-component case where the coefficients $w_1, w_2, \ldots$ of the dressing operator are 2×2-matrices. We consider the collection

$$\mathcal{A} = \{\ a_n\partial^n\ ,\ a_n \in \{1, \sigma_3\}\ ,\ n \in \mathbb{N}\ \} \tag{3.18}$$

of commuting bare operators, where the matrix coefficients a_n are either the 2×2 identity matrix or the third Pauli matrix $\sigma_3 = diag(1, -1)$. We review the arguments of [16] and show that the equations (3.8) generated by the bare operators $A = \sigma_3\partial$ and $B = \sigma_3\partial^2$ imply the Davey-Stewartson equation (in its "real" version).

Here, we shall use the zero-curvature formulation in order to derive the nonlinear equations in question. For convenience, however, the coefficient w_2 may be found directly in terms of w_1 by evaluating $W_y = -[W\sigma_3\partial W^{-1}]_{<0}W$. Its highest order reads

$$w_{1y} = [\sigma_3, w_2] + \sigma_3 w_{1x} + [w_1, \sigma_3]w_1 \quad . \tag{3.19}$$

Hence, we find the following parametrization of the differential operators

$$\begin{aligned} M(A) &= [W\sigma_3\partial W^{-1}]_{\geq 0} &= \sigma_3\partial + [w_1, \sigma_3] \ , \\ M(B) &= [W\sigma_3\partial^2 W^{-1}]_{\geq 0} &= \sigma_3\partial^2 + [w_1, \sigma_3]\partial - w_{1y} - \sigma_3 w_{1x} \end{aligned} \tag{3.20}$$

in terms of the field w_1. Now, we introduce

$$U = \begin{pmatrix} 0 & r \\ q & 0 \end{pmatrix} = [w_1, \sigma_3] \ ,\quad D = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \tag{3.21}$$

in order to decompose w_1 into its off-diagonal and diagonal part according to $w_1 = -\frac{1}{2}(\sigma_3 U + D)$. The zero-curvature condition $M(A)_t - M(B)_y = [M(B), M(A)]$ then leads to the evolution equation

$$2U_t = \sigma_3(U_{xx} + U_{yy} + UD_x + D_xU) + [D_y, U] + D_{yy} - D_{xx} + (U^2)_x + \sigma_3(U^2)_y \quad . \tag{3.22}$$

From the diagonal part of (3.19) one has

$$\sigma_3 D_y - D_x + U^2 = 0 \quad , \tag{3.23}$$

so that $D_{yy} - D_{xx} + (U^2)_x + \sigma_3(U^2)_y = 0$ and the diagonal part of (3.22) is satisfied identically. The combined system (3.22)/(3.23) may be rewritten as

$$U_t = \tfrac{1}{2}\sigma_3(U_{xx} + U_{yy}) + \sigma_3 U^3 + [D_y, U] \quad , \quad \sigma_3 D_y - D_x + U^2 = 0 \quad , \tag{3.24}$$

i.e.

$$\begin{aligned} r_t &= \tfrac{1}{2}(r_{xx} + r_{yy}) + r^2 q + (a_y - b_y) r \quad , & a_y - a_x + rq &= 0 \quad , \\ q_t &= -\tfrac{1}{2}(q_{xx} + q_{yy}) - q^2 r - (a_y - b_y) q \quad , & b_y + b_x - rq &= 0 \quad , \end{aligned} \tag{3.25}$$

which represents as 2+1-dimensional version of the first member of the AKNS hierarchy. As we have $(a+b)_y = (a-b)_x$ we may introduce the field χ by $a + b = \chi_x$, $a - b = \chi_y$, which has to satisfy $\chi_{xx} - \chi_{yy} = 2rq$. If we replace t by $-it$ and impose the constraints $q = \bar{r}$, $\chi = \bar{\chi}$ (the bar meaning complex conjugation), then this systems reduces to the Davey-Stewartson equation

$$ir_t = \tfrac{1}{2}(r_{xx} + r_{yy}) + \tfrac{1}{2} r(\chi_{xx} + \chi_{yy}) \quad , \quad \chi_{xx} - \chi_{yy} = 2|r|^2 \quad . \tag{3.26}$$

Commuting flows of higher order may be generated by using bare operators of the form ∂^n or $\sigma_3\partial^m$ as indicated in the previous section. The resulting hierarchy of evolution equations constitutes a 2+1-dimensional generalization of the "positive", i.e. the Taylor part of the AKNS system. It is the purpose of this paper to show that the part involving negative powers of the spectral parameter (and the corresponding Loewner system as its 2+1-dimensional extension) represents a squared eigenfunction symmetry of the positive hierarchy.

3.2 The modified KP hierarchy: Examples

3.2.1 The scalar case

Analogous to the case of the scalar KP hierarchy we shall now derive the (scalar) modified KP equation and its higher flows by considering a scalar dressing operator W. Again, the first two bare operators are chosen to be $A = \partial^2, B = \partial^3$. The evolution equations (3.8, k=1) then imply

$$\begin{aligned} w_{0y} &= w_{0xx} - 2w_{0x}^2 w_0^{-1} + 2(w_1 w_0^{-1})_x w_0 \quad , \\ w_{1y} &= w_{1xx} - 2w_{1x} w_{0x} w_0^{-1} + 2(w_2 w_0^{-1})_x w_0 \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} w_{0t} = \; & w_{0xxx} + 6w_{0x}^3 w_0^{-2} - 6w_{0x} w_{0xx} w_0^{-1} - 3w_{0x} w_0^{-1}(w_1 w_0^{-1})_x + 3(w_1 w_0^{-1})_{xx} \\ & -3w_1 w_0^{-1}(w_1 w_0^{-1})_x + 3(w_2 w_0^{-1})_x \quad . \end{aligned} \tag{3.28}$$

On use of (3.27) the fields w_1, w_2 can be expressed in terms of w_0 and its x- and y-derivatives. Insertion into (3.28) gives an equation for w_0 only. The analysis is simplified by introducing the new fields u, v_1, v_2 via

$$w_0 = e^u \quad , \quad w_1 = v_1\, e^u \quad , \quad w_2 = v_2\, e^u \quad . \tag{3.29}$$

Elimination of v_1 and v_2 as indicated above results in the (potential form of the) modified KP equation

$$4u_{tx} - u_{xxxx} + 6u_x^2 u_{xx} + 6u_{xx}u_y - 3u_{yy} = 0 \quad . \tag{3.30}$$

The associated scattering operators are readily calculated as

$$\begin{aligned} M(A) &= [W\partial^2 W^{-1}]_{\geq 1} &= \partial^2 - 2u_x\partial \quad , \\ M(B) &= [W\partial^3 W^{-1}]_{\geq 1} &= \partial^3 - 3u_x\partial^2 - \tfrac{3}{2}\,(u_{xx} - u_x^2 + u_y)\partial \end{aligned} \tag{3.31}$$

after substitution for w_1 and w_2.

The higher modified KP flows are obtained by "solving" the evolution $W_y = -[W\partial^2 W^{-1}]_{<1}W$ for the auxiliary fields $w_1, w_2, \ldots$ in terms of w_0. Insertion into the remaining equations $W_{t_n} = -[W\partial^n W^{-1}]_{<1}W$, $n \geq 3$ then yields the hierarchy of evolution equations for $u = \ln w_0$.

3.2.2 The 2×2 case

Here, as in the KP case, we consider the collection (3.18) of undressed operators. As pointed out in Ref. [16], the equations (3.8) generated by the bare operators $A = \sigma_3\partial$ and $B = \sigma_3\partial^2$ contain the Ishimori equation. To see this we note that the evolution equation (3.8) implies

$$w_{0y} = [S, w_1 w_0^{-1}]w_0 + S w_{0x} \quad , \quad S = w_0\sigma_3 w_0^{-1} \ . \tag{3.32}$$

The differential operators $M(A) = [W\sigma_3\partial W^{-1}]_{\geq 1}$ and $M(B) = [W\sigma_3\partial^2 W^{-1}]_{\geq 1}$ are calculated as

$$M(A) = S\partial \quad , \quad M(B) = S\partial^2 + T\partial \quad , \tag{3.33}$$

where

$$T = [w_1 w_0^{-1}, S] - 2S w_{0x} w_0^{-1} \overset{(3.32)}{=} - w_{0y} w_0^{-1} - S w_{0x} w_0^{-1} \quad . \tag{3.34}$$

The zero-curvature equation $M(A)_t - M(B)_y = [M(B), M(A)]$ leads to

$$S_y + SS_x = [S, T] \quad , \quad S_t - T_y = SS_{xx} + TS_x - ST_x \quad . \tag{3.35}$$

A more compact formulation of these equations is obtained by a representation

$$S = \vec{S}\vec{\sigma} = S_1\,\sigma_1 + S_2\,\sigma_2 + S_3\,\sigma_3 \quad , \quad T = \vec{T}\vec{\sigma} + \beta\,1 = T_1\,\sigma_1 + T_2\,\sigma_2 + T_3\,\sigma_3 + \beta\,1 \quad , \tag{3.36}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.37}$$

are the three Pauli matrices. We note that $S = w_0\sigma_3 w_0^{-1}$ is traceless and satisfies $S^2 = 1$. From (3.32) one finds

$$\mathrm{tr}\,(w_0^{-1}w_{0y}) = \mathrm{tr}\,(w_0^{-1}w_{0x}\sigma_3) \quad , \quad \mathrm{tr}\,(w_0^{-1}w_{0x}) = \mathrm{tr}\,(w_0^{-1}w_{0y}\sigma_3) \ . \tag{3.38}$$

With $\vec{S} = (S_1, S_2, S_3)$ and $\vec{T} = (T_1, T_2, T_3)$ one concludes

$$\beta = \frac{1}{2}\mathrm{tr}\,(T) = -\mathrm{tr}\,(w_0^{-1}w_{0y}) = -\tau_y \quad , \quad \vec{S}\vec{T} = \frac{1}{2}\mathrm{tr}\,(ST) = -\mathrm{tr}\,(w_0^{-1}w_{0x}) = -\tau_x \quad , \tag{3.39}$$

where $\tau = \ln\det w_0$. Decomposition of the matrix equations (3.35) yields

$$\vec{S}_y + i\vec{S}\times\vec{S}_x = 2i\,\vec{S}\times\vec{T} \tag{3.40}$$

and

$$\begin{aligned} \vec{S}_t - \vec{T}_y &= i\,\vec{S}\times\vec{S}_{xx} + i\,\vec{T}\times\vec{S}_x - i\,\vec{S}\times\vec{T}_x + \beta\,\vec{S}_x - \beta_x\,\vec{S} \quad , \\ -\beta_y &= \vec{S}\vec{S}_{xx} + \vec{T}\vec{S}_x - \vec{T}_x\vec{S} \quad . \end{aligned} \tag{3.41}$$

Since $\vec{T}$ is completely determined by (3.39) and (3.40), one can solve for $\vec{T}$ and obtains

$$\vec{T} = \frac{i}{2}\vec{S}\times\vec{S}_y + \frac{1}{2}\vec{S}_x - \tau_x\vec{S} \quad , \tag{3.42}$$

having used $\vec{S}^2 = 1$. Insertion into (3.41) yields the evolution equation

$$\vec{S}_t = \frac{i}{2}\vec{S}\times(\vec{S}_{xx} + \vec{S}_{yy}) - \tau_x\vec{S}_y - \tau_y\vec{S}_x \tag{3.43}$$

together with the hyperbolic equation

$$\tau_{xx} - \tau_{yy} = i\vec{S}(\vec{S}_x\times\vec{S}_y) \tag{3.44}$$

for the "auxiliary" field τ. These two equations represent the Ishimori system which is connected to the Davey-Stewartson equation via a Miura-type transformation. A particular reduction of a higher order symmetry of this system will be considered in Section 8.

4 The Symmetry Generated by Squared Eigenfunctions

4.1 Results for the KP hierarchy

We motivate the following considerations by reviewing the well-known result (e.g. [34]) that the products of eigenfunctions and adjoint eigenfunctions generate symmetries of the KP equation. With the scattering operators (3.17) derived in Section 3.1.1 one considers (adjoint) eigenfunctions ϕ_n, ψ_n ($n = 1, ..., N$) satisfying

$$\begin{aligned} \phi_{ny} &= \phi_{nxx} - 2w_{1x}\phi_n \quad , \quad \phi_{nt} = \phi_{nxxx} - 3w_{1x}\phi_{nx} - \tfrac{3}{2}(w_{1xx} + w_{1y})\phi_n \quad , \\ \psi_{ny} &= -\psi_{nxx} + 2w_{1x}\psi_n \quad , \quad \psi_{nt} = \psi_{nxxx} - 3w_{1x}\psi_{nx} - \tfrac{3}{2}(w_{1xx} - w_{1y})\psi_n \quad . \end{aligned} \tag{4.1}$$

It is easily verified that the infinitesimal transformation

$$w_1 \longrightarrow \tilde{w}_1 = w_1 + z\sum_{n=1}^{N}\phi_n\psi_n \tag{4.2}$$

is a symmetry transformation of the potential KP equation (3.16), i.e. for any solution w_1 the transformed field $\tilde{w}_1$ still satisfies (3.16) up to quadratic terms in the parameter z.

This result not only extends to the higher KP flows, when the (adjoint) eigenfunctions satisfy the time evolutions given by the corresponding higher differential operators $M(\partial^j) = [W\partial^j W^{-1}]_{\geq 0}$. Also the multicomponent KP hierarchy admits such a "squared eigenfunction symmetry", which we characterize by its infinitesimal generator, i.e. by the evolution equation with respect to the group parameter z:

Theorem 1: (The squared eigenfunction symmetry of the KP hierarchy)

Let the dressing operator W (3.5,k=0) satisfy the KP hierarchy (3.8,k=0) for some collection $\mathcal{A}$ of commuting bare operators. Let ϕ_n, ψ_n ($n = 1, ..., N$) be a set of (adjoint) eigenfunctions satisfying the hierarchy of commuting linear equations

$$\phi_{nt_A} = [M(A)\phi_n]_0 \quad , \quad \psi_{nt_A} = -[M(A)^* \psi_n]_0 \quad , \quad n = 1, ..., N \quad , \quad A \in \mathcal{A} \tag{4.3}$$

with $M(A) = [WAW^{-1}]_{\geq 0}$. Then the evolution equation

$$W_z = \sum_{n=1}^{N} \phi_n \partial^{-1} \psi_n^\dagger \;\; W \tag{4.4}$$

is compatible with the KP hierarchy. The differential operators $M(A)$ satisfy the "zero-curvature equations"

$$M(A)_z - (\sum_{n=1}^{N} \phi_n \partial^{-1} \psi_n^\dagger \;)_{t_A} = [\; \sum_{n=1}^{N} \phi_n \partial^{-1} \psi_n^\dagger \; , M(A)] \;\; , \tag{4.5}$$

which is equivalent to

$$M(A)_z = [\; \sum_{n=1}^{N} \phi_n \partial^{-1} \psi_n^\dagger \; , M(A)]_{\geq 0} \;\; . \tag{4.6}$$

The proof is given in the appendix (see also [29], where this result is stated in terms of the Lax operator instead of the dressing operator). We remark that both vector (column) or matrix solutions of the linear problems (4.3) can be used to generate the symmetry (4.4)/(4.6), where in the first case the "squared eigenfunction operator" represents a matrix valued pseudo-differential symbol with coefficients $(\; \sum_n \phi_n \partial^{-1} \psi_n^\dagger \;)_{ab} = \sum_n (\phi_n)_a \partial^{-1} (\psi_n)_b$.

Remark 1:

The symmetry generated by squared eigenfunctions exists independent of the special construction $M(A) = [WAW^{-1}]_{\geq 0}$ of the zero-curvature operators via a dressing formalism. In general, if a nonlinear equation is encoded as the zero-curvature condition of linear operators, then it is easily shown that the flow (4.6) leaves the zero-curvature equation invariant. Hence, outside the context of the KP hierarchy, the general squared eigenfunction symmetry is characterized by (4.6). In the KP context the linear operators $M(A)$ are parametrized in a special way in terms of the dressing operator. This parametrization is left invariant by this symmetry, as the transformation is given by the deformation (4.4) of the dressing operator itself.

The symmetry (4.2) of the scalar KP equation is a trivial corollary of Theorem 1. With $W = 1 + w_1 \partial^{-1} + ...$ the highest differential order of (4.4) yields

$$w_{1z} = \sum_{n=1}^{N} \; \phi_n \psi_n^\dagger \;\; , \tag{4.7}$$

which clearly generates the infinitesimal symmetry (4.2). The transpositions of the adjoint eigenfunctions are of course irrelevant in the scalar case.

We can now show that – as claimed in Section 2 – the system (2.9) is a symmetry of the multicomponent KP hierarchy:

Remark 2:

In the construction of the multicomponent KP hierarchy one associates the "time" variable $y = t_A$ with the undressed operator $A = S\partial$ with some constant matrix S, which commutes with the other bare operators used to construct the higher equations. The corresponding linear operator is $M(A) = [WAW^{-1}]_{\geq 0} = S\partial + [w_1, S]$, so that with $P = [w_1, S]$ the squared eigenfunction symmetry (4.7) *leads to* (2.9) *.*

4.2 Results for the modified KP hierarchy

The special feature of the differential operators $M(A) = [WAW^{-1}]_{\geq 1}$ occuring in the construction of the modified equations is the absence of the zero-order term. The right-hand side of the flow (4.6), however, will in general not be tangent to $M(A)$, as a zero-order term is present. Hence, a modified symmetry has to be constructed which preserves the special form of the zero-curvature operators.

One key fact is that one should use "integrated" adjoint eigenfunctions instead of proper eigenfunctions. We observe that due to the absence of a zero-order term the operators $M(A)\partial^{-1}$ and $\partial^{-1}M(A)^* = -(M(A)\partial^{-1})^*$ are still differential operators. In this situation we may integrate adjoint eigenfunctions and consider fields Φ, ω satisfying the linear problems

$$\Phi_{t_A} = [M(A)\Phi]_0 \quad , \quad \omega_{t_A} = -[\partial^{-1}M(A)^*\omega_x]_0 \equiv res(\partial^{-1}\omega_x^{\dagger}M(A)\partial^{-1})^{\dagger} \quad . \tag{4.8}$$

Evidently, $\Psi = \omega_x$ yields a proper adjoint eigenfunction, which, however, will not be used in the context of the modified KP hierarchy. Hence, with some abuse of notation, solutions Φ and ω of (4.8) will be referred to as eigenfunctions and adjoint eigenfunctions.

In analogy to the results for the KP hierarchy pairs of (adjoint) eigenfunctions generate a one-parameter symmetry group of the modified KP hierarchy (3.8,k=1). The parameter will again be called z, the group is given as the flow of the evolution equation in the following theorem.

Theorem 2: (The squared eigenfunction symmetry of the modified KP hierarchy)

Let the dressing operator W (3.5,k=1) *satisfy the modified KP hierarchy* (3.8,k=1) *for some collection $\mathcal{A}$ of commuting bare operators. Let ϕ_n, ω_n ($n = 1, ..., N$) be a set of (adjoint) eigenfunctions satisfying the hierarchy of commuting linear equations*

$$\phi_{nt_A} = [M(A)\phi_n]_0 \quad , \quad \omega_{nt_A} = -[\partial^{-1}M(A)^*\partial\omega_n]_0 \quad , \quad n = 1, ..., N \quad , \quad A \in \mathcal{A} \tag{4.9}$$

with $M(A) = [WAW^{-1}]_{\geq 1}$. Then the evolution equation

$$W_z = - \sum_{n=1}^{N} \phi_n\partial^{-1}\omega_n^{\dagger}\partial \;\; W \tag{4.10}$$

is compatible with the modified KP hierarchy. The differential operators $M(A)$ satisfy the "zero-curvature equations"

$$M(A)_z + (\sum_{n=1}^{N} \phi_n \partial^{-1} \omega_n^\dagger \partial)_{t_A} = -[\sum_{n=1}^{N} \phi_n \partial^{-1} \omega_n^\dagger \partial \ , M(A)] \ , \tag{4.11}$$

which is equivalent to

$$M(A)_z = -[\sum_{n=1}^{N} \phi_n \partial^{-1} \omega_n^\dagger \partial \ , M(A)]_{\geq 1} \ . \tag{4.12}$$

The proof is given in the appendix. As a simple corollary we derive the squared eigenfunction symmetry of the scalar modified KP equation (3.30). With the scattering operators (3.31) one considers (adjoint) eigenfunctions ϕ_n, ω_n $(n = 1, ..., N)$ satisfying

$$\begin{array}{lll} \phi_{ny} = \phi_{nxx} - 2u_x \phi_{nx} & , & \phi_{nt} = \phi_{nxxx} - 3u_x \phi_{nxx} + \frac{3}{2}(u_x^2 - u_{xx} - u_y)\phi_{nx} \ , \\ \omega_{ny} = -\omega_{nxx} - 2u_x \omega_{nx} & , & \omega_{nt} = \omega_{nxxx} + 3u_x \omega_{nxx} + \frac{3}{2}(u_x^2 + u_{xx} - u_y)\omega_{nx} \ . \end{array} \tag{4.13}$$

With $W = w_0 + w_1 \partial^{-1} + ... = e^u(1 + v_1 \partial^{-1} + ...)$ the highest differential order of (4.10) yields

$$u_z = -\sum_{n=1}^{N} \ \phi_i \omega_i^\dagger \ , \tag{4.14}$$

which generates an infinitesimal symmetry of the modified KP equation (3.30). The transpositions of the adjoint eigenfunctions are of course redundant in the scalar case.

We can now show that – as claimed in Section 2 – the system (2.10) is a symmetry of the multicomponent modified KP hierarchy:

Remark 3:

In the construction of the multicomponent hierarchy one associates the "time" variable $y = t_A$ with the undressed operator $A = \sigma \partial$ with some constant matrix σ, which commutes with the other bare operators used to construct the higher equations. The corresponding linear operator is $M(A) = [WAW^{-1}]_{\geq 1} = w_0 \sigma w_0^{-1} \partial$. In general the squared eigenfunction symmetry yields the equation

$$w_{0z} = -\sum_{n=1}^{N} \ \phi_n \omega_n^\dagger \ w_0 \tag{4.15}$$

for the leading coefficient of the dressing operator. Hence, with $S = w_0 \sigma w_0^{-1}$ the symmetry $S_z = [w_{0z} w_0^{-1}, S]$ is characterized by (2.10) up to the irrelevant substitution $z \to -z$.

5 Linear Representation of the Squared Eigenfunction Symmetry

To each equation $W_{t_A} = [WAW^{-1}]_{\geq k} W$ of the (modified) KP hierarchy there is a corresponding linear representation $\Phi_{t_A} = [M(A)\Phi]_0$, which encodes the nonlinear equations via compatibility. It turns out that also the flow W_z generated by squared eigenfunctions admits a linear representation. However, this linear equation for the eigenfunction is not a differential equation but involves an integration.

In order to establish the necessary notation we first observe the important fact that squared eigenfunctions can be integrated in terms of a potential introduced in the following Lemma 1. We consider an arbitrary differential operator $M(A)$, i.e. a polynomial in $\partial = \partial/\partial x$, where A denotes some index (the result is independent of the dressing construction, in which A represents the bare operator). One considers (adjoint) eigenfunctions Φ and Ψ, which may be regarded either as vectors (columns) or as matrices. In the following the integral $\Omega(\Psi, \Phi) = \int^x \Psi^\dagger \Phi$ will play a crucial role, where the time dependence of the integration constant has to be fixed in a suitable manner. For this we note that the transposition of the operator $M(A)$ is defined in such a way that for all functions Ψ and Φ the expressions $\Psi^\dagger [M(A)\Phi]_0$ and $[M(A)^*\Psi]_0^\dagger \Phi$ coincide up to "boundary terms", i.e. one has

$$\Psi^\dagger [M(A)\Phi]_0 - [M(A)^*\Psi]_0^\dagger \Phi = R(\Psi, \Phi)_x \quad , \tag{5.1}$$

where $R(\Psi, \Phi)_x$ is some bi-linear differential expression of its two arguments. It turns out that in terms of pseudo-differential symbols this expression admits the simple representation $R(\Psi, \Phi) = res(\partial^{-1}\Psi^\dagger M(A)\Phi\partial^{-1})$. With this observation one arrives at

Lemma 1: (The squared eigenfunction potential)

For any matrix differential operator $M(A)$ and any pair of (adjoint) eigenfunctions Φ and Ψ satisfying $\Phi_{t_A} = [M(A)\Phi]_0$, $\Psi_{t_A} = -[M(A)^*\Psi]_0$, there exists a potential $\Omega(\Psi, \Phi)$ satisfying

$$\Omega(\Psi, \Phi)_x = \Psi^\dagger \Phi \quad , \quad \Omega(\Psi, \Phi)_{t_A} = res(\partial^{-1}\Psi^\dagger M(A)\Phi\partial^{-1}) \quad . \tag{5.2}$$

The compatibility of these two equations is readily checked, as

$$(\Omega(\Psi, \Phi)_{t_A})_x = res(\partial^{-1}\Psi^\dagger M(A)\Phi\partial^{-1})_x = \Psi^\dagger [M(A)\Phi]_0 - [M(A)^*\Psi]_0^\dagger \Phi = (\Psi^\dagger \Phi)_{t_A} \quad . \tag{5.3}$$

If $M(A)$ represents a collection of operators labeled by $A \in \mathcal{A}$, then all linear equations may be considered simultaneously, if the operators satisfy zero-curvature equations. In this case it is verified (see e.g. [29]) that the t_A-evolutions (5.2) with different labels are compatible. The potential is defined up to an integration constant and may be regarded as a contour integration of the corresponding expressions in the (x, t_A)-plane. If both eigenfunctions and adjoint eigenfunctions are column vectors, then the corresponding Ω is a scalar function. For matrices Φ and Ψ the potential is a matrix field. Also mixed cases resulting in column or row valued Ω may be considered.

5.1 Results for the KP hierarchy

By means of the potential associated with squared eigenfunctions a linear representation for the symmetry (4.4) can be formulated.

Theorem 3: (The linear representation of the squared eigenfunction symmetry (KP))

Let W be a solution of the KP hierarchy (3.8,k=0), let ϕ_n, ψ_n ($n=1,...,N$) be a set of (adjoint) eigenfunctions (4.3) generating the symmetry (4.4)/(4.6). Let Φ and Ψ be an additional pair of (adjoint) eigenfunctions, let $\Omega(\psi_n,\Phi)$ and $\Omega(\Psi,\phi_n)$ be the potentials defined by (5.2). Then the scattering problems

$$\Phi_{t_A} = [M(A)\Phi]_0 \quad , \quad \Psi_{t_A} = -[M(A)^*\Psi]_0 \tag{5.4}$$

are compatible with the evolutions

$$\Phi_z = \sum_{n=1}^{N} \phi_n \Omega(\psi_n,\Phi) \quad , \quad \Psi_z = \sum_{n=1}^{N} \psi_n \Omega(\Psi,\phi_n)^\dagger \quad . \tag{5.5}$$

The proof is a straightforward calculation given in the appendix. The transposition of the potential in (5.5) is redundant, when vector valued (adjoint) eigenfunctions are considered for which the potential is a scalar quantity. According to this theorem the squared eigenfunction symmetry (4.6) is the compatibility condition of the linear problems (5.4) encoding the KP hierarchy and the linear integro-differential equations (5.5).

Remark 4:

We note that a pair Φ,Ψ of (adjoint) eigenfunctions satisfying (5.4) as well as (5.5) gives rise to an "extended" bi-linear potential $\Omega(\Psi,\Phi)$ defined by the compatible equations

$$\Omega(\Psi,\Phi)_x = \Psi^\dagger\Phi \quad , \quad \Omega(\Psi,\Phi)_{t_A} = res(\partial^{-1}\Psi^\dagger M(A)\Phi\partial^{-1}) \quad , \quad \Omega(\Psi,\Phi)_z = \sum_{n=1}^{N} \Omega(\Psi,\phi_n)\Omega(\psi_n,\Phi) \quad . \tag{5.6}$$

The compatibility of these equations is easily checked by a direct computation.

We note that we have not imposed a z-evolution on the (adjoint) eigenfunctions ϕ_n and ψ_n triggering the symmetry (4.4)/(4.6). The z-dependence of these fields is determined via compatibility and is a generalization of (5.5). Indeed, in the infinitesimal transformation characterized by $M(A)_z = [\,\sum_n \phi_n\partial^{-1}\psi_n^\dagger\,, M(A)]_{\geq 0}$, $\phi_{jt_A} = [M(A)\phi_j]_0$, $\psi_{it_A} = -[M(A)^*\psi_i]_0$, the time-evolutions of the (adjoint) eigenfunctions are assumed for all values of z, so that

$$(\phi_{jt_A})_z = ([M(A)\phi_j]_0)_z \quad , \quad (\psi_{it_A})_z = -([M(A)^*\psi_i]_0)_z \tag{5.7}$$

is implied. Inserting (4.6) into these equations one finds that

$$\begin{aligned} \sigma_j &:= \phi_{jz} - \sum_{n=1}^{N} \phi_n\Omega(\psi_n,\phi_j) \quad , \\ \gamma_i &:= \psi_{iz} - \sum_{n=1}^{N} \psi_n\Omega(\psi_i,\phi_n)^\dagger \quad , \\ \omega_{ij} &:= \Omega(\psi_i,\phi_j)_z - \sum_{n=1}^{N} \Omega(\psi_i,\phi_n)\Omega(\psi_n,\phi_j) \end{aligned} \tag{5.8}$$

satisfy

$$\begin{aligned} \sigma_{jt_A} &= [M(A)\sigma_j]_0 \quad , \\ \gamma_{it_A} &= -[M(A)^*\gamma_i]_0 \quad , \\ \omega_{ijx} &= \psi_i^\dagger\sigma_j + \gamma_i^\dagger\phi_j \quad , \\ \omega_{ijt_A} &= res(\partial^{-1}\psi_i^\dagger M(A)\sigma_j\partial^{-1}) + res(\partial^{-1}\gamma_i^\dagger M(A)\phi_j\partial^{-1}) \quad . \end{aligned} \tag{5.9}$$

Hence, σ_i, γ_j are (adjoint) eigenfunctions and ω_{ij} can be identified with $\Omega(\psi_i,\sigma_j)+\Omega(\gamma_i,\phi_j)$. The resulting z-evolutions of the triggering (adjoint) eigenfunctions and their potentials are thus given by

$$\begin{aligned} \phi_{jz} &= \sum_{n=1}^N \phi_n\Omega(\psi_n,\phi_j) \ + \ \sigma_j \quad , \\ \psi_{iz} &= \sum_{n=1}^N \psi_n\Omega(\psi_i,\phi_n)^\dagger \ + \ \gamma_i \quad , \\ \Omega(\psi_i,\phi_j)_z &= \sum_{n=1}^N \Omega(\psi_i,\phi_n)\Omega(\psi_n,\phi_j) + \Omega(\psi_i,\sigma_j) + \Omega(\gamma_i,\phi_j) \quad , \end{aligned} \tag{5.10}$$

where σ_j, γ_i are arbitrary (adjoint) eigenfunctions of the KP hierarchy. One may consistently assume $\sigma_j = \gamma_i = 0$, so that in this special reduction the squared eigenfunction symmetry reduces to the system of *ordinary* differential equations

$$\begin{aligned} \phi_{jz} &= \sum_{n=1}^N \phi_n\Omega(\psi_n,\phi_j) \quad , \\ \psi_{iz} &= \sum_{n=1}^N \psi_n\Omega(\psi_i,\phi_n)^\dagger \quad , \\ \Omega(\psi_i,\phi_j)_z &= \sum_{n=1}^N \Omega(\psi_i,\phi_n)\Omega(\psi_n,\phi_j) \quad , \\ M(A)_z &= [\sum_{n=1}^N \phi_n\partial^{-1}\psi_n^\dagger, M(A)]_{\geq 0} \quad , \end{aligned} \tag{5.11}$$

which can be integrated explicitly. In this situation the flow $M(z=0)\to M(z=\infty)$ turns out to be a (binary) Darboux transformation triggered by the (adjoint) eigenfunctions. Details are to be published in a subsequent paper.

5.2 Results for the modified KP hierarchy

A corresponding result holds for the modified hierarchy. Here integrated adjoint eigenfunctions ω are considered, which give rise to a potential via the proper adjoint eigenfunctions $\Psi=\omega_x$, i.e. one has to introduce the potential $\Omega(\omega_x,\Phi)$ defined by

$$\Omega(\omega_x,\Phi)_x = \omega_x^\dagger\Phi \quad , \quad \Omega(\omega_x,\Phi)_{t_A} = res(\partial^{-1}\omega_x^\dagger M(A)\Phi\partial^{-1}) \quad . \tag{5.12}$$

One obtains the squared eigenfunction symmetry (4.12) as the compatibility condition of the following linear problems:

Theorem 4: (The linear representation of the squared eigenfunction symmetry (modified KP))

Let W be a solution of the modified KP hierarchy (3.8,k=1), let ϕ_n, ω_n ($n = 1, ..., N$) be a set of (adjoint) eigenfunctions (4.9) generating the symmetry (4.10)/(4.12). Let Φ and ω be an additional pair of (adjoint) eigenfunctions, let $\Omega(\omega_{nx}, \Phi)$ and $\Omega(\omega_x, \phi_n)$ be the potentials defined by (5.12). Then the scattering problems

$$\Phi_{t_A} = [M(A)\Phi]_0 \quad , \quad \omega_{t_A} = -[\partial^{-1} M(A)^* \partial \omega]_0 \tag{5.13}$$

are compatible with the evolutions

$$\Phi_z = \sum_{n=1}^{N} \phi_n (\Omega(\omega_{nx}, \Phi) - \omega_n^\dagger \Phi) \quad , \quad \omega_z = \sum_{n=1}^{N} \omega_n \Omega(\omega_x, \phi_n)^\dagger \quad . \tag{5.14}$$

The proof is a straightforward calculation given in the appendix.

We note that in analogy to the results for the KP hierarchy the z-evolutions (5.14) of a pair of (adjoint) eigenfunctions yield a z-evolution of the corresponding potential. The result formally coincides with (5.6), i.e. for a pair ω, Φ satisfying (5.13) as well as (5.14) there exists an "extended" bi-linear potential $\Omega(\omega_x, \Phi)$ defined by the compatible equations

$$\Omega(\omega_x, \Phi)_x = \omega_x^\dagger \Phi \quad , \quad \Omega(\omega_x, \Phi)_{t_A} = res(\partial^{-1} \omega_x^\dagger M(A) \Phi \partial^{-1}) \quad , \quad \Omega(\omega_x, \Phi)_z = \sum_{n=1}^{N} \Omega(\omega_x, \phi_n)\, \Omega(\omega_{nx}, \Phi) \quad . \tag{5.15}$$

The compatibility of these equations is checked by a direct computation.

As in the KP case the z-dependence of the triggering (adjoint) eigenfunctions is a generalization of (5.14). In the infinitesimal transformation characterized by $M(A)_z = -[\ \sum_n \phi_n \partial^{-1} \omega_n^\dagger \partial \ , M(A)]_{\geq 1}$, $\phi_{jt_A} = [M(A)\phi_j]_0$, $\omega_{it_A} = -[\partial^{-1} M(A)^* \partial \omega_i]_0$, the time-evolutions of the (adjoint) eigenfunctions are assumed for all values of z, so that

$$(\phi_{jt_A})_z = ([M(A)\phi_j]_0)_z \quad , \quad (\omega_{it_A})_z = -([\partial^{-1} M(A)^* \partial \omega_i])_z \tag{5.16}$$

is implied. Inserting (4.12) into these equations one finds that

$$\begin{aligned} \sigma_j &:= \phi_{jz} - \sum_{n=1}^{N} \phi_n (\Omega(\omega_{nx}, \phi_j) - \omega_n^\dagger \phi_j) \quad , \\ \gamma_i &:= \omega_{iz} - \sum_{n=1}^{N} \omega_n \Omega(\omega_{ix}, \phi_n)^\dagger \quad , \\ \omega_{ij} &:= \Omega(\omega_{ix}, \phi_j)_z - \sum_{n=1}^{N} \Omega(\omega_{ix}, \phi_n) \Omega(\omega_{nx}, \phi_j) \end{aligned} \tag{5.17}$$

satisfy

$$\begin{aligned} \sigma_{jt_A} &= [M(A)\sigma_j]_0 \quad , \\ \gamma_{it_A} &= -[\partial^{-1} M(A)^* \partial \gamma_i]_0 \quad , \\ \omega_{ijx} &= \omega_{ix}^\dagger \sigma_j + \gamma_{ix}^\dagger \phi_j \quad , \\ \omega_{ijt_A} &= res(\partial^{-1} \omega_{ix}^\dagger M(A) \sigma_i \partial^{-1}) + res(\partial^{-1} \gamma_{ix}^\dagger M(A) \phi_j \partial^{-1}) \quad . \end{aligned} \tag{5.18}$$

Hence, σ_i, γ_j are (adjoint) eigenfunctions and ω_{ij} can be identified with $\Omega(\omega_{ix},\sigma_j)+\Omega(\gamma_{ix},\phi_j)$. The resulting z-evolutions of the triggering (adjoint) eigenfunctions and their potentials are thus given by

$$\begin{aligned} \phi_{jz} &= \sum_{n=1}^{N}\phi_n(\Omega(\omega_{nx},\phi_j)-\omega_n^\dagger\phi_j) \ + \ \sigma_j \quad , \\ \omega_{iz} &= \sum_{n=1}^{N}\omega_n\Omega(\omega_{nx},\phi_i)^\dagger \ + \ \gamma_i \quad , \\ \Omega(\omega_{ix},\phi_j)_z &= \sum_{n=1}^{N}\Omega(\omega_{ix},\phi_n)\Omega(\omega_{nx},\phi_j)+\Omega(\omega_{ix},\sigma_j)+\Omega(\gamma_{ix},\phi_j) \quad , \end{aligned} \tag{5.19}$$

where σ_j, γ_i are arbitrary (adjoint) eigenfunctions of the modified KP hierarchy. One may consistently assume $\sigma_j=\gamma_i=0$, so that in this special reduction the squared eigenfunction symmetry reduces to the system of *ordinary* differential equations

$$\begin{aligned} \phi_{jz} &= \sum_{n=1}^{N}\phi_n(\Omega(\omega_{nx},\phi_j)-\omega_n^\dagger\phi_j) \quad , \\ \omega_{iz} &= \sum_{n=1}^{N}\omega_n\Omega(\omega_{ix},\phi_n)^\dagger \quad , \\ \Omega(\omega_{ix},\phi_j)_z &= \sum_{n=1}^{N}\Omega(\omega_{ix},\phi_n)\Omega(\omega_{nx},\phi_j) \quad , \\ M(A)_z &= -[\sum_{n=1}^{N}\phi_n\partial^{-1}\omega_n^\dagger\partial, M(A)]_{\geq 1} \quad , \end{aligned} \tag{5.20}$$

which can be integrated explicitly. As in the KP case, the flow $M(z=0)\to M(z=\infty)$ turns out to generate a (binary) Darboux transformation triggered by the (adjoint) eigenfunctions.

6 Scattering Problems of Loewner Type

We now show that the Loewner system as reviewed in Section 2 is a special squared eigenfunction symmetry of the (modified) KP hierarchy. The idea simply is that via differentiation w.r.t. x the integro-differential linear problems (5.5)/(5.14) can be converted into a purely differential problem of higher order. We note that in the modified case the Loewner system (2.7) corresponds to the symmetry (4.10)/(4.12) after the irrelevant substitution $z\to -z$.

6.1 Results for the KP hierarchy

Differentiation of (5.5) with only one pair ϕ_1,ψ_1, i.e.

$$\Phi_z=\phi_1\Omega(\psi_1,\Phi) \tag{6.1}$$

leads to

$$\Phi_{zx}=\phi_{1x}\Omega(\psi_1,\Phi)+\phi_1\psi_1^\dagger\Phi \ , \tag{6.2}$$

so that elimination of the potential via (6.1) yields the second order problem

$$\Phi_{zx}=\phi_{1x}\phi_1^{-1}\,\Phi_z+\phi_1\psi_1^\dagger\Phi \quad . \tag{6.3}$$

If we consider an undressed operator $A = S\partial$ with a constant matrix S , one finds the scattering operator $M(A) = [WAW^{-1}]_{\geq 0} = S\partial + [w_1, S]$ from the dressing construction of the KP hierarchy. With $P := [w_1, S]$ and the associated "time" $t_A = y$ one obtains the scattering problem

$$\Phi_y = S\,\Phi_x + P\,\Phi \quad , \tag{6.4}$$

which after differentiation of (6.1) yields

$$\begin{aligned} \Phi_{zy} &= \phi_{1y}\phi_1^{-1}\,\Phi_z + \phi_1 res(\partial^{-1}\psi_1^\dagger(S\partial + P)\Phi\partial^{-1}) \\ &= \phi_{1y}\phi_1^{-1}\,\Phi_z + \phi_1\psi_1^\dagger S\,\Phi \quad . \end{aligned} \tag{6.5}$$

The linear problems (6.4), (6.3) and (6.5) represent the Loewner triad (2.5). Thus, the Loewner system (2.4) has been identified as the squared eigenfunction symmetry of the KP hierarchy generated by a single pair ϕ_1, ψ_1 of (adjoint) eigenfunctions.

The general symmetry (4.4) generated by an arbitrary number of pairs ϕ_n, ψ_n $(n = 1, ..., N)$ may also be characterized by a purely differential linear problem, where (5.5) has to be differentiated N times and the potentials are eliminated using the equations for Φ_z, Φ_{zx} etc.

6.2 Results for the modified KP hierarchy

We consider the special case where two pairs ω_1, ϕ_1 and ω_2, ϕ_2 are used to generate the symmetry (4.10). The linear problem

$$\Phi_z = \phi_1\Omega(\omega_{1x}, \Phi) + \phi_2\Omega(\omega_{2x}, \Phi) - (\phi_1\omega_1^\dagger + \phi_2\omega_2^\dagger)\Phi \tag{6.6}$$

of Theorem 4 yields after differentiation

$$\Phi_{zx} = \phi_{1x}\Omega(\omega_{1x}, \Phi) + \phi_{2x}\Omega(\omega_{2x}, \Phi) - (\phi_{1x}\omega_1^\dagger + \phi_{2x}\omega_2^\dagger)\Phi - (\phi_1\omega_1^\dagger + \phi_2\omega_2^\dagger)\Phi_x \quad . \tag{6.7}$$

The modified hierarchy is essentially characterized by the fact that the scattering operators have no zero-order terms. Consequently, the linear problems (4.9) admit the trivial solutions $\phi_n \equiv 1$ and $\omega_n \equiv 1$. For the choice $\omega_2 \equiv 1$, $\Omega(\omega_{2x}, \Phi) = 0$ one can use (6.6) to express $\Omega(\omega_{1x}, \Phi)$ as a differential expression of Φ. Insertion into (6.7) yields a purely differential equation of second order:

$$\Phi_{zx} = \phi_{1x}\phi_1^{-1}\,\Phi_z - (\phi_1\omega_1^\dagger + \phi_2)\,\Phi_x + (\phi_{1x}\phi_1^{-1}\phi_2 - \phi_{2x})\,\Phi \quad . \tag{6.8}$$

If we consider an undressed operator $A = \sigma\partial$ with a constant matrix σ , one finds the scattering operator $M(A) = [WAW^{-1}]_{\geq 1} = w_0\sigma w_0^{-1}\partial$ from the dressing construction of the modified KP hierarchy. With $S := w_0\sigma w_0^{-1}$ and the associated "time" $t_A = y$ one obtains the scattering problem

$$\Phi_y = S\,\Phi_x \quad , \tag{6.9}$$

which after differentiation of (6.6) yields

$$\begin{aligned} \Phi_{zy} &= \phi_{1y}\phi_1^{-1}\,(\Phi_z + \phi_2\Phi) + \phi_1 res(\partial^{-1}\omega_{1x}^\dagger S\partial\Phi\partial^{-1}) - \phi_1(\omega_1^\dagger\Phi)_y - (\phi_2\Phi)_y \\ &= \phi_{1y}\phi_1^{-1}\,\Phi_z - (\phi_1\omega_1^\dagger + \phi_2)\,\Phi_y + (\phi_{1y}\phi_1^{-1}\phi_2 - \phi_{2y})\,\Phi \quad . \end{aligned} \tag{6.10}$$

The linear problems (6.9), (6.8) and (6.10) represent the Loewner triad (2.8) up to the irrelevant substitution $z \to -z$. Thus, the Loewner system (2.7) has been identified as the squared eigenfunction symmetry of the modified KP hierarchy generated by "$1\frac{1}{2}$" pairs ϕ_1, ω_1 and $\phi_2, \omega_2 \equiv 1$ of (adjoint) eigenfunctions.

7 Reductions

7.1 Results for the KP hierarchy

The KP hierarchy (3.8) admits certain constraints, which may conveniently be described in terms of the dressing operator. For example, the so-called BKP hierarchy [8] is obtained from the KP hierarchy via a constraint of this type. These reductions were used by Kupershmidt in Ref. [23] and consequently were termed "Kupershmidt-constraints" in Ref. [17]. They are given as relations between the zero-curvature operators $M(A)$ and their adjoints.

Theorem 5: (Constraints of the KP hierarchy)

Let B be a constant matrix. Then the constraint

$$W^*BW = B \tag{7.1}$$

is left invariant by all those evolution equations of the KP hierarchy (3.8,k=0), *for which the undressed operator A satisfies*

$$A^*B + BA = 0 \quad . \tag{7.2}$$

The corresponding differential operator $M(A) = [WAW^{-1}]_{\geq 0}$ satisfies

$$M(A)^*B + BM(A) = 0 \quad . \tag{7.3}$$

For the proof one derives the identity

$$(W^*BW)_{t_A} = -W^*[W^{-1*}\,(A^*B + BA)\,W^{-1}]_{<0}W \quad , \tag{7.4}$$

so that the KP flows subject to (7.2) are tangent to the constraint (7.1). For the differential operators one finds $M(A)^*B + BM(A)$= $[W^{-1*}(A^*B + BA)W^{-1}]_{\geq 0}$=0.

We note that if the matrix B is not symmetric or skew-symmetric, then (7.1) actually represents two constraints (via transposition of (7.1)). We also remark that the constraint may be rephrased as $W^{-1*}BW^{-1} = B$ or (for invertible B) as $WB^{-1}W^* = B^{-1}$. From here a more general formulation

$$[WCW^*]_{<0} = 0 \quad \text{or} \quad [W^{-1*}DW^{-1}]_{<0} = 0 \tag{7.5}$$

of the constraint is possible, where C and D may be any constant operators satisfying $AC + CA^* = 0$ and $DA + A^*D = 0$ respectively. Indeed, one finds the identities

$$(WCW^*)_{t_A} = -W\,(AC + CA^*)\,W^* + [WAW^{-1}]_{\geq 0}WCW^* + WCW^*[W^{-1*}A^*W^*]_{\geq 0} \tag{7.6}$$

and

$$
\begin{aligned}
(W^{-1*}DW^{-1})_{t_A} &= W^{-1*}\,(A^*D+DA)\,W^{-1} \\
&\quad - [W^{-1*}A^*W^*]_{\geq 0}W^{-1*}DW^{-1} - W^{-1*}DW^{-1}[WAW^{-1}]_{\geq 0} \quad ,
\end{aligned}
\tag{7.7}
$$

so that from (7.6) the tangency condition $([WCW^*]_{<0})_{t_A} = -[W\,(AC+CA^*)\,W^*]_{<0} = 0$ follows, whereas (7.7) yields $([W^{-1*}DW^{-1}]_{<0})_{t_A} = [W^{-1*}\,(A^*D+DA)\,W^{-1}]_{<0} = 0$.

Also the squared eigenfunction symmetry can preserve the constraint (7.1)/(7.3):

Remark 5:

The squared eigenfunction symmetry

$$W_z = M'W \quad , \quad M(A)_z = [M', M(A)]_{\geq 0} \tag{7.8}$$

with $M' = \sum_n \phi_n \partial^{-1}\psi_n^\dagger$ *leaves the constraint* (7.1)/ (7.3) *invariant if*

$$M'^*B + BM' = 0 \quad . \tag{7.9}$$

This is easily verified observing $(W^*BW)_z = W^*(M'^*B + BM')W$. From the relation $M(A)^*B = -BM(A)$ between the scattering operators and their adjoints one finds a relation between the eigenfunctions and the adjoint eigenfunctions: for any eigenfunction Φ the field $\Psi = B\Phi$ is an adjoint eigenfunction. This may be used to satisfy (7.9) and hence to preserve the constraint (7.1)/(7.3). For example, for symmetric B one can consider a collection of eigenfunctions ϕ_n and associated adjoint eigenfunctions $\psi_n = B\phi_n$. The corresponding operator

$$M' = \sum_{n=1}^{N} \phi_n \partial^{-1}\phi_n^\dagger B \tag{7.10}$$

satisfies (7.9). Alternatively, for invertible B one can consider a collection of (adjoint) eigenfunctions ϕ_n, ψ_n and generate the symmetry with

$$M' = \sum_{n=1}^{N} \phi_n \partial^{-1}\psi_n^\dagger + \sum_{n=1}^{N} B^{-1}\psi_n \partial^{-1}\phi_n^\dagger B \quad , \tag{7.11}$$

which again satisfies (7.9) for symmetric or skew-symmetric B.

7.2 Results for the modified KP hierarchy

In the modified case the analogous results are given by

Theorem 6: (Constraints of the modified KP hierarchy)

Let B and C be constant matrices. Then the constraint

$$W^* B\partial W = C\partial \tag{7.12}$$

is left invariant by all those evolution equations of the modified KP hierarchy (3.8,k=1)*, for which the undressed operator A satisfies*

$$A^* C\partial + C\partial A = 0 \quad . \tag{7.13}$$

The corresponding differential operator $M(A) = [WAW^{-1}]_{\geq 1}$ satisfies

$$M(A)^* B\partial + B\partial\, M(A) = 0 \quad . \tag{7.14}$$

The proof is given in the appendix. For the modified constraint (7.12) two matrices B and C can be chosen, whereas in the KP case (7.1) these matrices must coincide (the leading order of the dressing operator is 1). We note that the transformation $W \to \tilde{W} = Wa$ with any constant matrix $a \in \mathcal{A}$ leaves the modified KP hierarchy (3.8,k=1) invariant. This gauge freedom is reflected by the fact that the two matrices in (7.12) may be chosen independently. Under the transformation the constraint becomes $\tilde{W}^* B\partial\tilde{W} = a^* C a\partial$, so that for fixed B the choices C and $a^* C a$ lead to the same reduced hierarchy.

We note that if the matrices B and C are not symmetric (or skew-symmetric), then (7.12) actually represents two constraints (via transposition of (7.12)).

We also note that the constraint implies $[W^{-1*} C\partial W^{-1}]_{<1} = 0$. In this form a more general formulation

$$[W^{-1*} D W^{-1}]_{<1} = 0 \tag{7.15}$$

of the constraint is possible, where D may be any constant operator satisfying $DA + A^* D = 0$. One finds

$$\begin{aligned}
(W^{-1*} D W^{-1})_{t_A} &= \partial\,[\partial^{-1} W^{-1*} A^* W^*]_{<0} W^{-1*} D W^{-1} \;+\; W^{-1*} D W^{-1} [WAW^{-1}]_{<1} \\
&= W^{-1*}\; (A^* D + DA)\; W^{-1} \\
&\quad -\; \partial\,[\partial^{-1} W^{-1*} A^* W^*]_{\geq 0} W^{-1*} D W^{-1} \;-\; W^{-1*} D W^{-1} [WAW^{-1}]_{\geq 1} \;,
\end{aligned} \tag{7.16}$$

so that $([W^{-1*} D W^{-1}]_{<1})_{t_A} = [W^{-1*}\; (A^* D + DA)\; W^{-1}]_{<1} = 0$.

Also the squared eigenfunction symmetry can preserve the constraint (7.12)/(7.14):

Remark 6:

The squared eigenfunction symmetry

$$W_z = -M' W \quad , \quad M(A)_z = -[M', M(A)]_{\geq 1} \tag{7.17}$$

with $M' = \sum_n \phi_n \partial^{-1} \omega_n^\dagger \partial$ *leaves the constraint* (7.12)/(7.14) *invariant if*

$$M'^* B\partial + B\partial M' = 0 \quad . \tag{7.18}$$

This is easily verified with $(W^* B\partial W)_z = -W^*(M'^* B\partial + B\partial M')W$. From $\partial^{-1} M(A)^* \partial B = -BM(A)$ one finds a relation between the eigenfunctions and the (integrated) adjoint eigenfunctions: for any eigenfunction Φ the field $\omega = B\Phi$ is an adjoint eigenfunction. This may be used to satisfy (7.18) and hence to preserve the constraint (7.12)/(7.14). For example, for skew-symmetric B one can consider a collection of eigenfunctions ϕ_n and associated adjoint eigenfunctions $\omega_n = B\phi_n$. The corresponding operator

$$M' = \sum_{n=1}^{N} \phi_n \partial^{-1} \phi_n^\dagger B\partial \tag{7.19}$$

satisfies (7.18). For symmetric or skew-symmetric B one can consider a collection of (adjoint) eigenfunctions ϕ_n, ω_n and use

$$M' = \sum_{n=1}^{N} \phi_n \partial^{-1} \omega_n^\dagger \partial - \sum_{n=1}^{N} B^{-1} \omega_n \partial^{-1} \phi_n^\dagger B\partial \quad , \tag{7.20}$$

which again satisfies (7.18).

8 Examples

In Section 3, the construction of the (modified) KP hierarchy has been illustrated by way of several examples. Here, their squared eigenfunction symmetries will be derived explicitly. In particular, we shall establish the compatibility of the modified Nizhnik-Veselov-Novikov equation [14] and the 2+1-dimensional sine-Gordon equation [18]. This generalizes the well-known fact that the potential modified Korteweg-de Vries equation

$$\varphi_t = \varphi_{xxx} + \tfrac{1}{2}\varphi_x^3 \tag{8.1}$$

and the sine-Gordon equation

$$\varphi_{zx} = \sin\varphi \tag{8.2}$$

can be solved simultaneously.

8.1 The KP hierarchy

8.1.1 The scalar case

In order to formulate the simplest non-trivial squared eigenfunction symmetry of the scalar KP equation (3.16) it is sufficient to deal with only one pair of (adjoint) eigenfunctions ψ, ϕ. The z-evolution (4.4), i.e. $W_z = \phi\partial^{-1}\psi\, W$, supplemented by the y-evolution (4.3) of the (adjoint) eigenfunctions produces the system

$$\begin{aligned} w_{1z} &= \phi\psi \;, \\ \phi_y &= \phi_{xx} - 2w_{1x}\phi \;, \\ \psi_y &= -\psi_{xx} + 2w_{1x}\psi \;. \end{aligned} \tag{8.3}$$

Interestingly, in the 1+1-dimensional limit with respect to ∂_y and $\psi = \phi$ the above system reduces to a single equation for w_1 only. Thus, if we assume that nothing depends on y we obtain $\phi^2 = w_{1z}$ and

$$w_{1zxx} = \tfrac{1}{2}\frac{w_{1zx}^2}{w_{1z}} + 4w_{1x}w_{1z} \tag{8.4}$$

which is, by construction, compatible with the potential Korteweg-de Vries equation

$$w_{1t} = \tfrac{1}{4}w_{1xxx} - \tfrac{3}{2}w_{1x}^2 \;. \tag{8.5}$$

8.1.2 The 2×2 case

In Section 3.3.2 it has been shown that the first member of the 2+1-dimensional AKNS hierarchy (3.25) is given in terms of the lowest order of the dressing operator W, viz

$$w_1 = -\frac{1}{4}\begin{pmatrix} \chi_y + \chi_x & 2r \\ -2q & \chi_y - \chi_x \end{pmatrix} \;. \tag{8.6}$$

Hence, the residue of the z-evolution (4.4) for one pair of (adjoint) eigenfunctions ψ, ϕ

$$W_z = \phi\partial^{-1}\psi^{\dagger}W \tag{8.7}$$

is given by

$$\begin{aligned} r_z &= -2\phi_1\psi_2 \quad , \quad \chi_{xz} = -2(\phi_1\psi_1 + \phi_2\psi_2) \;, \\ q_z &= 2\phi_2\psi_1 \quad\;\; , \quad \chi_{yz} = -2(\phi_1\psi_1 - \phi_2\psi_2) \;, \end{aligned} \tag{8.8}$$

where the subscripts denote the vector components of the (adjoint) eigenfunctions. These evolve according to

$$\begin{aligned} \phi_y &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\phi_x + \begin{pmatrix} 0 & r \\ q & 0 \end{pmatrix}\phi \;, \\ \psi_y &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\psi_x - \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}\psi \;. \end{aligned} \tag{8.9}$$

Thus, (8.8) and (8.9) are compatible with the AKNS equation (3.25) together with the corresponding t-evolution for the (adjoint) eigenfunctions.

Remarkably, this squared eigenfunction symmetry has a distinct meaning in the context of classical geometry. In fact, on setting

$$\begin{array}{lllll} \beta_{12} = r & , & \beta_{32} = \phi_1 & , & \beta_{23} = \psi_1 \ , \\ \beta_{21} = q & , & \beta_{31} = \phi_2 & , & \beta_{13} = -\psi_2 \ , \\ \partial_1 = \partial_y - \partial_x & , & \partial_2 = \partial_y + \partial_x & , & \partial_3 = \frac{1}{2}\partial_z \ , \end{array} \tag{8.10}$$

(8.8), (8.9) transform into

$$\beta_{ik,l} = \beta_{il}\beta_{lk} \tag{8.11}$$

together with

$$\chi_{,3,i} = -2\beta_{i3}\beta_{3i} \quad , \quad i \neq 3 \tag{8.12}$$

for distinct values of i, k and l. The system (8.11) was investigated by Darboux at the turn of the century [7, 12]. It is descriptive of conjugate coordinate systems given in terms of the "rotation coefficients" β_{ik}. The "Darboux system" is of current interest since, in particular, it contains the 2+1-dimensional integrable sine-Gordon equation

$$\left(\frac{\theta_{zx}}{\sin\theta}\right)_x - \left(\frac{\theta_{zy}}{\sin\theta}\right)_y + \frac{\theta_x\theta_{zy} - \theta_y\theta_{zx}}{\sin^2\theta} = 0 \quad , \tag{8.13}$$

as recently introduced and discussed in Refs. [21, 38].

8.2 The modified KP hierarchy

8.2.1 The scalar case

The simplest non-trivial squared eigenfunction symmetry of the scalar modified KP equation (3.30) is obtained by the choice $\phi_1 = \phi$, $\omega_1 = 1$, $\phi_2 = 1$ and $\omega_2 = \omega$. In this case, the z-evolution (4.12) of $M(\partial^2)$ simplifies to

$$M_z(\partial^2) = -[\phi + \partial^{-1}\omega\partial, M(\partial^2)]_{\geq 1} \quad . \tag{8.14}$$

Hence, with (3.31) and the y-evolution (4.9) of the (adjoint) eigenfunctions we deduce

$$\begin{aligned} v_z &= -(\phi + \omega)_x \quad , \\ \phi_y &= \phi_{xx} - 2v\phi_x \quad , \\ \omega_y &= -\omega_{xx} - 2v\omega_x \end{aligned} \tag{8.15}$$

with $v = u_x$. The above system may now be transformed into a single non-local equation for one quantity only. For this purpose it is convenient to introduce the quantities φ and α according to

$$e^\varphi = \phi_x \quad , \quad \alpha = \phi_y \quad . \tag{8.16}$$

Conversely, these equations may be regarded as definition of the potential ϕ provided the integrability condition

$$\alpha_x = (e^\varphi)_y \tag{8.17}$$

is satisfied. Finally, we solve the system (8.15) for ω_x, ω_y and v to obtain

$$\begin{aligned}
v &= \tfrac{1}{2}\varphi_x - \tfrac{1}{2}\alpha e^{-\varphi} \quad , \\
\omega_x &= -\tfrac{1}{2}\varphi_{zx} - e^{\varphi} + \tfrac{1}{2}(\alpha e^{-\varphi})_z \quad , \\
\omega_y &= \tfrac{1}{2}[e^{\varphi}(\varphi_{zx} + 2e^{\varphi})]_x e^{-\varphi} + \tfrac{1}{4}(\alpha^2 e^{-2\varphi})_z - \tfrac{1}{2}\varphi_{yz} - \alpha \quad ,
\end{aligned} \tag{8.18}$$

the integrability condition of which yields

$$([e^{\varphi}(\varphi_{zx} + 2e^{\varphi})]_x e^{-\varphi})_x + \tfrac{1}{2}(\alpha^2 e^{-2\varphi})_{zx} - (\alpha e^{-\varphi})_{zy} = 0 \quad . \tag{8.19}$$

The nonlinear equation (8.19) with $\alpha = \partial_x^{-1}(e^{\varphi})_y$ represents a 2+1-dimensional generalization of the sinh/cosh-Gordon or Liouville equation as pointed out in Ref. [3]. To see this we perform the limit $v_y = \phi_y = \omega_y = 0$ from which we conclude that $\alpha = 0$. Thus, (8.19) can be integrated where the integration constant has to vanish due to (8.18). Further integration then leads to

$$\varphi_{zx} + 2e^{\varphi} + f(z)e^{-\varphi} = 0 \tag{8.20}$$

where the function f may be transformed into a constant.

8.2.2 The 2×2 case

The Loewner system (2.7) with $\phi_1 = 1, \omega_1 = \omega, \phi_2 = \phi$ constitutes the simplest squared eigenfunction symmetry of the 2×2 modified KP hierarchy. Hence, if we demand that $S^2 = 1$ the Loewner system

$$\begin{aligned}
S_z &= [\phi + \omega^{\dagger}, S] \quad , \\
\phi_y &= S\phi_x \quad , \\
\omega_y^{\dagger} &= \omega_x^{\dagger} S
\end{aligned} \tag{8.21}$$

is compatible with the Ishimori hierarchy the first member of which is given by (3.35). The explicit link may be seen best by introducing the vector notation (in analogy to (3.36))

$$\begin{aligned}
S &= \vec{S}\vec{\sigma} \quad , \\
\phi + \omega^{\dagger} &= \vec{a}\vec{\sigma} + \nu 1 \quad , \\
\phi - \omega^{\dagger} &= \vec{b}\vec{\sigma} + \sigma 1 \quad ,
\end{aligned} \tag{8.22}$$

as used in the derivation of (3.43). In these quantities (8.21) translates into

$$\begin{aligned}
S_z &= 2i\vec{a} \times \vec{S} \quad , \\
\vec{b}_x &= i\vec{S} \times \vec{a}_y + \sigma_y \vec{S} \quad , \\
\vec{b}_y &= i\vec{S} \times \vec{a}_x + \sigma_x \vec{S} \quad ,
\end{aligned} \tag{8.23}$$

and

$$\begin{aligned}
\nu_x = \vec{S}\vec{a}_y \quad &, \quad \sigma_x = \vec{S}\vec{b}_y \quad , \\
\nu_y = \vec{S}\vec{a}_x \quad &, \quad \sigma_y = \vec{S}\vec{b}_x \quad .
\end{aligned} \tag{8.24}$$

The latter relations represent the trace terms of (8.21) and (8.21) multiplied by S respectively.

The first of the equations (8.23) determines $\vec{a}$ up to its component in $\vec{S}$-direction, viz

$$\vec{a} = \frac{1}{2i}\vec{S} \times \vec{S}_z + \rho\vec{S} \quad . \tag{8.25}$$

The compatibility condition for (8.24) with respect to ν then produces

$$[\rho_x + \frac{1}{2i}\vec{S}(\vec{S}_x \times \vec{S}_z)]_x = [\rho_y + \frac{1}{2i}\vec{S}(\vec{S}_y \times \vec{S}_z)]_y \quad . \tag{8.26}$$

Finally, cross-differentiation of the remaining equations of (8.23) leads to

$$[\vec{S}(\vec{S}\vec{S}_{zx}) - \vec{S}_{zx} + 2i\rho\vec{S} \times \vec{S}_x + 2\sigma_x\vec{S}]_x = [\vec{S}(\vec{S}\vec{S}_{zy}) - \vec{S}_{zy} + 2i\rho\vec{S} \times \vec{S}_y + 2\sigma_y\vec{S}]_y \quad . \tag{8.27}$$

The contribution of the above in $\vec{S}$-direction takes the simple form

$$\sigma_{xx} - \sigma_{yy} = \tfrac{1}{4}(\vec{S}_y^2 - \vec{S}_x^2)_z \quad . \tag{8.28}$$

For the complete compatibility of the Ishimori system (3.43), (3.44) and the Loewner system in the form (8.26)-(8.28) it remains to discuss two items:

Firstly, the Ishimori field τ needs to be identified in the Loewner system (8.26)-(8.28). To this end we consider the zero-order of the z-evolution

$$W_z = (\phi + \partial^{-1}\omega^\dagger\partial)W \quad . \tag{8.29}$$

Taking its trace we immediately arrive at

$$\tau_z = \operatorname{tr}(w_0^{-1}w_{0z}) = \operatorname{tr}(\phi + \omega^\dagger) = 2\nu \quad . \tag{8.30}$$

In fact, this is not surprising since on inserting (8.25) into (8.24) and calculating the compatibility condition with respect to ρ we end up with

$$\nu_{xx} - \nu_{yy} = \frac{i}{2}[\vec{S}(\vec{S}_x \times \vec{S}_y)]_z \quad , \tag{8.31}$$

which is nothing but the hyperbolic equation (3.44) in terms of ν. (Note that $\vec{S}_z(\vec{S}_x \times \vec{S}_y) = 0$!)

Secondly, the t-evolution of the auxiliary fields ρ and σ may be calculated from (4.9).

8.2.3 A reduction

As illustration of the reduction theory developed in Section 7 the final part of this section is devoted to the link between a 2+1-dimensional integrable sine-Gordon system [18] and the modified Nizhnik-Veselov-Novikov equation [14]. The sine-Gordon system is most conveniently obtained by specializing the matrix $S = \vec{S}\vec{\sigma}$ in the Loewner system (8.26)-(8.28) to

$$S = \begin{pmatrix} 0 & e^{-\varphi} \\ e^{\varphi} & 0 \end{pmatrix} \quad , \quad \text{i.e.} \quad \vec{S} = (2\cosh\varphi, \tfrac{2}{i}\sinh\varphi, 0) \quad . \tag{8.32}$$

The Loewner system (8.27) then assumes the simple form

$$\begin{aligned} \varphi_{xxz} - \varphi_{yyz} &= 2\sigma_x\varphi_x - 2\sigma_y\varphi_y \quad , \\ \sigma_{xx} - \sigma_{yy} &= \tfrac{1}{4}(\varphi_x^2 - \varphi_y^2)_z \end{aligned} \tag{8.33}$$

with $\rho = 0$.

Before we discuss the obvious question why the constraint (8.32) is admissible let us briefly take a closer look at the system (8.33). On the one hand, (8.33) may be brought into the form of a 2+1-dimensional sinh-Gordon system after the change of variables $\varphi = \gamma + \gamma'$ and $\sigma = -\frac{1}{2}(\gamma - \gamma')_z$. It becomes

$$\begin{aligned} (e^{\gamma+\gamma'}\gamma_{zx})_x &= (e^{\gamma+\gamma'}\gamma_{zy})_y \quad , \\ (e^{-\gamma-\gamma'}\gamma'_{zx})_x &= (e^{-\gamma-\gamma'}\gamma'_{zy})_y \end{aligned} \tag{8.34}$$

as set down in Ref. [18]. On the other hand, the connection to the alternative form

$$\begin{aligned} &\left(\frac{\theta_{zx}}{\sin\theta}\right)_x - \left(\frac{\theta_{zy}}{\sin\theta}\right)_y + \frac{\theta_x\theta'_y - \theta_y\theta'_x}{\sin^2\theta} = 0 \quad , \\ &\left(\frac{\theta'_x}{\sin\theta}\right)_x - \left(\frac{\theta'_y}{\sin\theta}\right)_y + \frac{\theta_x\theta_{zy} - \theta_y\theta_{zx}}{\sin^2\theta} = 0 \end{aligned} \tag{8.35}$$

is achieved by means of

$$\begin{aligned} \theta &= -i(\gamma + \gamma') \quad , \\ \theta'_x &= \theta_{zy}\cos\theta + (\gamma - \gamma')_{zy}\sin\theta \quad , \\ \theta'_y &= \theta_{zx}\cos\theta + (\gamma - \gamma')_{zx}\sin\theta \quad . \end{aligned} \tag{8.36}$$

It is observed that the sine-Gordon system (8.35) reduces to the scalar equation (8.13) for $\theta' = \theta_z$.

We note that the Ishimori equation (3.43)/(3.44) does not admit the specialization (8.32). Indeed, the sinh-Gordon system above is not compatible with the full Ishimori hierarchy, but only with part of it. More specifically, one has to invoke Theorem 6 in order to find a systematic interpretation of the constraint (8.32).

In fact, if we consider the differential operator of lowest order of the Ishimori hierarchy, viz

$$M(A) = S\partial \quad , \quad A = \sigma_3\partial \quad , \tag{8.37}$$

then it can easily be seen that the relations

$$M(A)^*\sigma_1\partial + \sigma_1\partial M(A) = 0 \quad , \quad A^*\sigma_3\partial + \sigma_3\partial A = 0 \tag{8.38}$$

hold with real $M(A)$. According to Theorem 6 we may impose the constraint

$$W^*\sigma_1\partial W = \sigma_3\partial \tag{8.39}$$

on the Ishimori hierarchy (and therefore the Loewner system) which implies (8.38). However, as the bare operator $\sigma_3\partial^2$ associated with the t-evolution (3.43) of the Ishimori equation does not satisfy the second relation of (8.38), i.e. (7.13) with $C = \sigma_3$, the Ishimori equation itself cannot be reduced to the special form (8.32). One might replace this bare operator by $i\sigma_3\partial^2$, which leads to an alternative version of the Ishimori equation. This, in turn, would not be in accord with a real valued dressing operator.

More generally, only the bare operators $(\sigma_3 + \epsilon 1)\partial^n$ of odd order obey (8.38) and meet the above-mentioned reality requirement. Consequently, the first bare operator of higher order being admissible is

$$A' = (\sigma_3 + \epsilon 1)\partial^3 \tag{8.40}$$

for constant ϵ. We shall briefly discuss the equation associated with (8.40).

It turns out that neither the zero-curvature formulation nor the dressing operator formalism prevents the calculation of the nonlinear equations associated with the operators A and A' from being lengthy and laborious. Thus, it seems to be appropriate to use the following mixed approach: From the definition

$$M(A') = (WA'W^{-1})_{\geq 1} \tag{8.41}$$

we find that $M(A')$ has the form

$$M(A') = U\partial + T\partial^2 + (S + \epsilon 1)\partial^3 \tag{8.42}$$

for $S = w_0\sigma_3 w_0^{-1}$. The zero-curvature condition then implies

$$\begin{aligned} T &= \begin{pmatrix} \frac{1}{2}\varphi_y + \frac{3\epsilon}{2}\varphi_x & -\varphi_x e^{-\varphi} \\ \varphi_x e^{\varphi} & -\frac{1}{2}\varphi_y - \frac{3\epsilon}{2}\varphi_x \end{pmatrix} , \\ U &= \begin{pmatrix} \frac{1}{2}\varphi_x\varphi_y + \frac{1}{4}\varphi_{xy} + \frac{3\epsilon}{4}\varphi_{xx} + \frac{3\epsilon}{4}\varphi_x^2 + \frac{1}{4}\chi_x & (-\varphi_{xx} - \frac{1}{4}\varphi_{yy} + \frac{1}{4}\varphi_y^2 - \frac{3\epsilon}{4}\varphi_{xy} + \frac{1}{4}\chi_y)e^{-\varphi} \\ (\varphi_{xx} + \frac{1}{4}\varphi_{yy} + \frac{1}{4}\varphi_y^2 + \frac{3\epsilon}{4}\varphi_{xy} + \frac{1}{4}\chi_y)e^{\varphi} & \frac{1}{2}\varphi_x\varphi_y - \frac{1}{4}\varphi_{xy} - \frac{3\epsilon}{4}\varphi_{xx} + \frac{3\epsilon}{4}\varphi_x^2 + \frac{1}{4}\chi_x \end{pmatrix} \end{aligned} \tag{8.43}$$

with

$$\begin{aligned} \chi_{xx} - \chi_{yy} + \tfrac{3}{2}(\partial_y + \epsilon\partial_x)(\varphi_x^2 - \varphi_y^2) = 0 \quad , \\ 4\varphi_s = \epsilon\varphi_{xxx} + \varphi_{yyy} + 3\epsilon\varphi_{xyy} + 3\varphi_{xxy} + \epsilon\varphi_x^3 + \varphi_y^3 + \varphi_x\chi_x + \varphi_y\chi_y \quad , \end{aligned} \tag{8.44}$$

where s is the time associated with A'. The above system (ϵ=1) constitutes the modified Nizhnik-Veselov-Novikov equation which is related to the well-known Nizhnik-Veselov-Novikov equation [28] via a Miura-type transformation. The latter is of considerable interest since it is one of those equations which contain the spatial variables on an equal basis and admit localized solutions (dromions) [2]. The modified Nizhnik-Veselov-Novikov equation may be regarded as a 2+1-dimensional generalization of the potential modified Korteweg-de Vries equation as can be seen by performing the limit $\varphi_y = \chi_y = 0$.

The first equation of the modified Nizhnik-Veselov-Novikov system (8.44) and the second equation of the sine-Gordon system in the form (8.33) suggest the identification

$$\chi_z = -6\ (\partial_y + \epsilon\partial_x)\sigma \quad . \tag{8.45}$$

Indeed, on the one hand, comparison of the two parametrizations of $M(A')$ (8.41) and (8.42) leads to

$$\chi = -6 \ \mathrm{tr}\,[(\sigma_3 + \epsilon 1) w_0^{-1} w_1] \quad , \tag{8.46}$$

where the y-evolution of the dressing operator W has been used to eliminate suitable quantities involving w_1. On the other hand, the residue of the z-evolution $W_z = (\phi + \partial\omega^\dagger\partial^{-1})W$ yields

$$\mathrm{tr}\,(w_0^{-1} w_1)_z = \sigma_x \quad , \quad \mathrm{tr}\,(\sigma_3 w_0^{-1} w_1)_z = \sigma_y \quad . \tag{8.47}$$

Combination of the last three relations then produce (8.45). The s-evolution of the auxiliary field σ (or χ) is encoded in the corresponding evolution of the (adjoint) eigenfunctions ϕ and ω.

To summarize: We have shown that the modified Nizhnik-Veselov-Novikov equation (8.44) is connected to the 2+1-dimensional sine-Gordon system (8.35) in the same manner as the potential Korteweg-de Vries equation to the classical sine-Gordon equation. In both cases the respective pair of equations can be solved simultaneously and therefore share common solutions.

9 Conclusions and perspectives

We have demonstrated that for a matrix differential operator M the "scattering" problem $\Phi_t = [M\Phi]_0$ and its adjoint $\Psi_t = [M^*\Psi]_0$ admit the symmetry transformation generated by

$$\Phi_z = \textstyle\sum_n \phi_n \Omega(\psi_n, \Phi) \quad , \quad \Psi_z = \sum_n \psi_n \Omega(\Psi, \Phi_n)^\dagger \quad , \quad M_z = [\sum_n \phi_n \partial^{-1} \psi_n^\dagger, M]_{\geq 0} \quad ,$$

which is triggered by pairs ϕ_n, ψ_n of (adjoint) eigenfunctions. If M represents a collection of operators, then the zero-curvature conditions between the operators are preserved under this transformation. In the context of soliton equations in 2+1 dimensions interesting sets of zero-curvature operators are obtained from dressing transformations leading to the multicomponent KP hierarchy. The squared eigenfunction symmetry corresponds to a transformation of the dressing operator and thus preserves the parametrization of the differential operators.

For operators without zero-order terms the symmetry can be modified to

$$\Phi_z = \textstyle\sum_n \phi_n (\Omega(\omega_{nx}, \Phi) - \omega_n^\dagger \Phi) \quad , \quad \omega_z = \sum_n \omega_n \Omega(\omega_x, \Phi_n)^\dagger \quad , \quad M_z = -[\sum_n \phi_n \partial^{-1} \omega_n^\dagger \partial, M]_{\geq 1} \quad ,$$

which preserves this special form. Here, integrated adjoint eigenfunctions are to be used to generate the transformation. Such operators naturally occur in the construction of the multicomponent modified KP hierarchy.

For the special squared eigenfunction symmetry generated by only one pair of (adjoint) eigenfunctions the linear representation of this symmetry may be reformulated as a scattering problem of Loewner type. In the modified case "$1\frac{1}{2}$" pairs may be considered, where one of the adjoint eigenfunctions is chosen as a constant.

These symmetries represent interesting equations in their own right. Thus, some of the recently discovered integrable systems in 2+1 dimensions connected with the Loewner triad are identified as squared eigenfunction symmetries of the (modified) KP hierarchy. This provides the answer to the natural quest for the hierarchy of "higher flows" for these systems. In particular, the Zakharov-Manakov (or Darboux) system may be regarded as a symmetry of the 2+1-dimensional AKNS hierarchy. The 2+1-dimensional sine-Gordon system of Konopelchenko and Rogers represents a symmetry of the Ishimori hierarchy, which may be reduced to the hierarchy of the modified Nizhnik-Veselov-Novikov equation.

An interesting connection of these symmetries with the concept of Darboux transformations is found. As demonstrated in Section 5, the squared eigenfunction symmetry may be reduced to a set of simple *ordinary* differential equations (5.11)/(5.20). It can be shown that the integration of these equations gives rise to the binary Darboux transformations [27]. A detailed analysis of this connection is to be published in a forthcoming paper.

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Appendix

We note that a list of identities is given in Ref. [29], which are useful for the following computations with pseudo-differential symbols.

Proof of Theorem 1:

For the compatibility of the evolution (4.4) with a member of the KP hierarchy (3.8,k=0) one calculates

$$\begin{aligned}(W_{t_A})_z = \quad \sum_n \Big(& -[\phi_n\partial^{-1}\psi_n^\dagger WAW^{-1}]_{<0}W + [WAW^{-1}\phi_n\partial^{-1}\psi_n^\dagger]_{<0}W \\ & -[WAW^{-1}]_{<0}\phi_n\partial^{-1}\psi_n^\dagger W \Big) \quad .\end{aligned}$$

With $WAW^{-1} = M(A) + [WAW^{-1}]_{<0}$ these terms may be rewritten as

$$\begin{aligned} & -[\phi_n\partial^{-1}\psi_n^\dagger M(A)]_{<0}W - [\phi_n\partial^{-1}\psi_n^\dagger[WAW^{-1}]_{<0}]_{<0}W + [M(A)\phi_n\partial^{-1}\psi_n^\dagger]_{<0}W \\ & +[\,[WAW^{-1}]_{<0}\phi_n\partial^{-1}\psi_n^\dagger]_{<0}W - [WAW^{-1}]_{<0}\phi_n\partial^{-1}\psi_n^\dagger W \\ = \quad & -[\phi_n\partial^{-1}\psi_n^\dagger M(A)]_{<0}W - \phi_n\partial^{-1}\psi_n^\dagger[WAW^{-1}]_{<0}W + [M(A)\phi_n\partial^{-1}\psi_n^\dagger]_{<0}W \quad .\end{aligned}$$

This yields

$$
\begin{aligned}
(W_z)_{t_A} - (W_{t_A})_z \;&=\; \sum_n \Big(\;\phi_{nt_A}\partial^{-1}\psi_n^\dagger + \phi_n\partial^{-1}\psi_{nt_A}^\dagger \\
&\qquad +[\phi_n\partial^{-1}\psi_n^\dagger M(A)]_{<0} - [M(A)\phi_n\partial^{-1}\psi_n^\dagger]_{<0}\Big)\; W\ .
\end{aligned}
\tag{A.1}
$$

The first and the last term cancel, as

$$
[M(A)\phi_n\partial^{-1}\psi_n^\dagger]_{<0} = [M(A)\phi_n]_0\partial^{-1}\psi_n^\dagger\ . \tag{A.2}
$$

We further observe with $res(operator) = -res(operator^*)^\dagger$ that

$$
\begin{aligned}
[\phi_n\partial^{-1}\psi_n^\dagger M(A)]_{<0} \;&=\; \phi_n\partial^{-1}res(\partial^{-1}\psi_n^\dagger M(A)) \;=\; \phi_n\partial^{-1}res(M(A)^*\psi_n\partial^{-1})^\dagger \\
&=\; \phi_n\partial^{-1}[M(A)^*\psi_n]_0^\dagger\ ,
\end{aligned}
\tag{A.3}
$$

where the identity

$$
[\partial^{-1}C]_{<0} = \partial^{-1}res(\partial^{-1}C) \tag{A.4}
$$

is used, which holds for arbitrary differential operators C. Hence, also the second and third term in the right-hand side of (A.1) cancel.

The z-evolution (4.6) follows directly from the definition of $M(A)$ as

$$
M(A)_z = [\ \sum_n \phi_n\partial^{-1}\psi_n^\dagger\ ,\ WAW^{-1}\ ]_{\geq 0} = [\ \sum_n \phi_n\partial^{-1}\psi_n^\dagger\ ,\ [WAW^{-1}]_{\geq 0}\ ]_{\geq 0}\ \ ,
$$

because the negative orders of WAW^{-1} do not contribute. Hence the positive orders of (4.5) are satisfied. Further,

$$
\begin{aligned}
[\ \sum_n \phi_n\partial^{-1}\psi_n^\dagger\ ,\ M(A)\ ]_{<0} \;&=\; \sum_n \Big(\phi_n[\partial^{-1}\psi_n^\dagger M(A)]_{<0} - [M(A)\phi_n\partial^{-1}\psi_n^\dagger]_{<0}\Big) \\
&\overset{(A.2),(A.3)}{=}\; \sum_n \Big(\phi_n\partial^{-1}[M(A)^*\psi_n]_0^\dagger - [M(A)\phi_n]_0\partial^{-1}\psi_n^\dagger\Big) \\
&=\; -\sum_n \Big(\phi_n\partial^{-1}\psi_n^\dagger\Big)_{t_A}\ \ ,
\end{aligned}
$$

so that also the negative orders of (4.5) are satisfied.

Proof of Theorem 2:

For the compatibility of the evolution (4.10) with a member of the modified KP hierarchy (3.8,k=1) one calculates

$$
\begin{aligned}
(W_{t_A})_z = \;\sum_n \Big(\; &[\phi_n\partial^{-1}\omega_n^\dagger\partial WAW^{-1}]_{<1}W - [WAW^{-1}\phi_n\partial^{-1}\omega_n^\dagger\partial]_{<1}W \\
&+[WAW^{-1}]_{<1}\phi_n\partial^{-1}\omega_n^\dagger\partial W\;\Big)\ \ .
\end{aligned}
$$

With $WAW^{-1} = M(A) + [WAW^{-1}]_{<1}$ these terms may be rewritten as

$$
\begin{aligned}
&[\phi_n\partial^{-1}\omega_n^\dagger\partial M(A)]_{<1}W + [\phi_n\partial^{-1}\omega_n^\dagger\partial[WAW^{-1}]_{<1}]_{<1}W - [M(A)\phi_n\partial^{-1}\omega_n^\dagger\partial]_{<1}W \\
&-[\ [WAW^{-1}]_{<1}\phi_n\partial^{-1}\omega_n^\dagger\partial]_{<1}W + [WAW^{-1}]_{<1}\phi_n\partial^{-1}\omega_n^\dagger\partial W \\
=\;\; &[\phi_n\partial^{-1}\omega_n^\dagger\partial M(A)]_{<1}W + \phi_n\partial^{-1}\omega_n^\dagger\partial[WAW^{-1}]_{<1}W - [M(A)\phi_n\partial^{-1}\omega_n^\dagger\partial]_{<1}\,W\ \ .
\end{aligned}
$$

This yields

$$(W_z)_{t_A} - (W_{t_A})_z = -\sum_n \Big(\phi_{nt_A}\partial^{-1}\omega_n^\dagger\partial + \phi_n\partial^{-1}\omega^\dagger_{nt_A}\partial + [\phi_n\partial^{-1}\omega_n^\dagger\partial M(A)]_{<1} - [M(A)\phi_n\partial^{-1}\omega_n^\dagger\partial]_{<1} \Big)\ W \ . \tag{A.5}$$

With $[M(A)\phi_n\partial^{-1}\omega_n^\dagger\partial]_{<1} = [M(A)\phi_n]_0\partial^{-1}\omega_n^\dagger\partial$ the first and the last term cancel (note that $[M(A)\phi_n]_{\geq 1}$ does not contribute to $[M(A)\phi_n\partial^{-1}\omega_n^\dagger\partial]_{<1}$). Further,

$$\begin{aligned} &[\phi_n\partial^{-1}\omega_n^\dagger\partial M(A)]_{<1}W = [\phi_n\omega_n^\dagger M(A)]_{<1}W - [\phi_n\partial^{-1}\omega_{nx}^\dagger M(A)]_{<1}W \\ &= -[\phi_n\partial^{-1}\omega_{nx}^\dagger M(A)]_{<1}W = -[\phi_n\partial^{-1}\omega_{nx}^\dagger M(A)\partial^{-1}]_{<0}\partial W = -\phi_n\partial^{-1}res(\partial^{-1}\omega_{nx}^\dagger M(A)\partial^{-1})\partial W \quad , \end{aligned}$$

where in the last step the identity (A.4) is used. The residue represents the time evolution of $\omega_n^\dagger$, so that also the second and the third term in the right-hand side of (A.5) cancel.

The z-evolution (4.12) follows directly from the definition of $M(A)$ as

$$M(A)_z = -[\ \textstyle\sum_n \phi_n\partial^{-1}\omega_n^\dagger\partial\ ,\ WAW^{-1}]_{\geq 1} = -[\ \sum_n \phi_n\partial^{-1}\omega_n^\dagger\partial\ ,\ [WAW^{-1}]_{\geq 1}]_{\geq 1} \ .$$

Hence the positive orders of (4.11) are satisfied. Further,

$$\begin{aligned} -[\ \textstyle\sum_n \phi_n\partial^{-1}\omega_n^\dagger\partial\ ,\ M(A)\ ]_{<1} &= -\textstyle\sum_n \Big([\phi_n\partial^{-1}\omega_n^\dagger\partial M(A)]_{<1} - [M(A)\phi_n\partial^{-1}\omega_n^\dagger\partial]_{<1}\Big) \\ &= \textstyle\sum_n \Big(-[\phi_n\omega_n^\dagger M(A)]_{<1} + [\phi_n\partial^{-1}\omega_{nx}^\dagger M(A)]_{<1} + [M(A)\phi_n]_0\partial^{-1}\omega_n^\dagger\partial \Big) \ . \end{aligned}$$

With $M(A) = [M(A)]_{\geq 1}$ the first term vanishes, whereas with (A.4) the second term yields

$$\textstyle\sum_n \phi_n[\partial^{-1}\omega_{nx}^\dagger M(A)\partial^{-1}]_{<0}\partial = \sum_n \phi_n\partial^{-1}res(\partial^{-1}\omega_{nx}^\dagger M(A)\partial^{-1})\partial = \sum_n \phi_n\partial^{-1}\omega_{nt_A}^\dagger\partial \ .$$

Hence, also the non-positive orders of (4.11) are satisfied.

Proof of Theorem 3:

The compatibility of (5.4) and (5.5) for the eigenfunction Φ follows from

$$\begin{aligned} (\Phi_{t_A})_z &= \textstyle\sum_n \Big([\ [\phi_n\partial^{-1}\psi_n^\dagger\ ,\ M(A)]_{\geq 0}\Phi\ ]_0 + [M(A)\phi_n\Omega(\psi_n,\Phi)]_0 \Big) \\ &= \textstyle\sum_n \Big([\ \phi_n\partial^{-1}\psi_n^\dagger M(A)\Phi\ ]_0 - [\ M(A)\phi_n\partial^{-1}\psi_n^\dagger\Phi\ ]_0 + [M(A)\phi_n\Omega(\psi_n,\Phi)]_0 \Big) \\ &= \textstyle\sum_n \Big(\phi_n res(\partial^{-1}\psi_n^\dagger M(A)\Phi\partial^{-1}) - [\ M(A)\phi_n\partial^{-1}\ (\partial\Omega(\psi_n,\Phi) - \Omega(\psi_n,\Phi)\partial)\ ]_0 \\ &\qquad + [M(A)\phi_n\Omega(\psi_n,\Phi)]_0 \Big) \\ &= \textstyle\sum_n \Big(\phi_n res(\partial^{-1}\psi_n^\dagger M(A)\Phi\partial^{-1}) + [\ M(A)\phi_n\partial^{-1}\Omega(\psi_n,\Phi)\partial\ ]_0 \Big) \\ &\overset{(*)}{=} \textstyle\sum_n \Big(\phi_n res(\partial^{-1}\psi_n^\dagger M(A)\Phi\partial^{-1}) + [M(A)\phi_n]_0[\partial^{-1}\Omega(\psi_n,\Phi)\partial]_0 \Big) \\ &= \textstyle\sum_n \Big(\phi_n\Omega(\psi_n,\Phi)_{t_A} + \phi_{nt_A}\Omega(\psi_n,\Phi) \Big) \ = \ (\Phi_z)_{t_A} \ . \end{aligned}$$

For $(*)$ we note that $[M(A)\Phi]_{\geq 1}$ does not contribute to the second term.

For the adjoint eigenfunctions $\Psi_{t_A} = -[M(A)^*\Psi]_0 = -res(M(A)^*\Psi\partial^{-1}) = -res(\partial^{-1}\Psi^\dagger M(A))^\dagger$ holds, so that one obtains

$$
\begin{aligned}
(\Psi^\dagger_{t_A})_z &\overset{(*)}{=} \sum_n \Big(-res(\partial^{-1}\Omega(\psi,\phi_n)\psi_n^\dagger M(A)) - res(\partial^{-1}\psi^\dagger[\phi_n\partial^{-1}\psi_n^\dagger, M(A)]_{\geq 0}) \Big) \\
&= \sum_n \Big(-res(\partial^{-1}\Omega(\psi,\phi_n)\psi_n^\dagger M(A)) - res(\partial^{-1}\psi^\dagger\phi_n\partial^{-1}\psi_n^\dagger M(A)) \\
&\qquad\qquad + res(\partial^{-1}\psi^\dagger M(A)\phi_n\partial^{-1}\psi_n^\dagger) \Big) \\
&= \sum_n \Big(-res(\partial^{-1}\Omega(\psi,\phi_n)\psi_n^\dagger M(A)) - res(\ \{\Omega(\psi,\phi_n)\partial^{-1} - \partial^{-1}\Omega(\psi,\phi_n)\}\ \psi_n^\dagger M(A)) \\
&\qquad\qquad + res(\partial^{-1}\psi^\dagger M(A)\phi_n\partial^{-1}\psi_n^\dagger) \Big) \\
&= \sum_n \Big(-\Omega(\psi,\phi_n) res(\partial^{-1}\psi_n^\dagger M(A)) + res(\partial^{-1}\psi^\dagger M(A)\phi_n\partial^{-1})\psi_n^\dagger \Big) \\
&= \sum_n \Big(\Omega(\psi,\phi_n)\psi^\dagger_{nt_A} + \Omega(\psi,\phi_n)_{t_A}\psi_n^\dagger \Big) \ = \ (\Psi_z^\dagger)_{t_A} \quad .
\end{aligned}
$$

For $(*)$ we note that $[\phi_n\partial^{-1}\psi_n^\dagger, M(A)]_{<0}$ does not contribute to the residue.

Proof of Theorem 4:

The compatibility of (5.13) and (5.14) for the eigenfunction Φ follows from

$$
\begin{aligned}
(\Phi_{t_A})_z &= ([M(A)\Phi]_0)_z \\
&= \sum_n \Big(-[\ [\phi_n\partial^{-1}\omega_n^\dagger\partial\ ,\ M(A)]_{\geq 1}\Phi\ ]_0 + [M(A)\phi_n(\Omega(\omega_{nx},\Phi) - \omega_n^\dagger\Phi)]_0 \Big) \\
&= \sum_n \Big([\ [-\phi_n\omega_n^\dagger + \phi_n\partial^{-1}\omega_{nx}^\dagger\ ,\ M(A)]_{\geq 1}\Phi\ ]_0 + [M(A)\phi_n(\Omega(\omega_{nx},\Phi) - \omega_n^\dagger\Phi)]_0 \Big) \\
&= \sum_n \Big([\ [-\phi_n\omega_n^\dagger + \phi_n\partial^{-1}\omega_{nx}^\dagger\ ,\ M(A)]_{\geq 0}\Phi\ ]_0 - [\ [-\phi_n\omega_n^\dagger + \phi_n\partial^{-1}\omega_{nx}^\dagger\ ,\ M(A)]_0\Phi\ ]_0 \\
&\qquad + [M(A)\phi_n(\Omega(\omega_{nx},\Phi) - \omega_n^\dagger\Phi)]_0 \Big) \\
&\overset{(*)}{=} \sum_n \Big(-\phi_n\omega_n^\dagger[M(A)\Phi]_0 + [\ \phi_n\partial^{-1}\omega_{nx}^\dagger M(A)\Phi\ ]_0 - [\ M(A)\phi_n\partial^{-1}\omega_{nx}^\dagger\Phi\ ]_0 \\
&\qquad - [M(A)\phi_n\omega_n^\dagger]_0\Phi - [\phi_n\partial^{-1}\omega_{nx}^\dagger M(A)]_0\Phi \\
&\qquad + [M(A)\phi_n\partial^{-1}\omega_{nx}^\dagger]_0\Phi + [M(A)\phi_n\Omega(\omega_{nx},\Phi)]_0 \Big) \\
&= \sum_n \Big(-\phi_n\omega_n^\dagger\Phi_{t_A} + \phi_n res(\partial^{-1}\omega_{nx}^\dagger M(A)\Phi\partial^{-1}) \\
&\qquad - [\ M(A)\phi_n\partial^{-1}\{\partial\Omega(\omega_{nx},\Phi) - \Omega(\omega_{nx},\Phi)\partial\}\ ]_0 - [M(A)\phi_n\omega_n^\dagger]_0\Phi \\
&\qquad - \phi_n res(\partial^{-1}\omega_{nx}^\dagger M(A)\partial^{-1})\Phi + [M(A)\phi_n\partial^{-1}\{\partial\omega_n^\dagger - \omega_n^\dagger\partial\}]_0\Phi \\
&\qquad + [M(A)\phi_n\Omega(\omega_{nx},\Phi)]_0 \Big) \\
&= \sum_n \Big(-\phi_n\omega_n^\dagger\Phi_{t_A} + \phi_n\Omega(\omega_{nx},\Phi)_{t_A} + [M(A)\phi_n\partial^{-1}\Omega(\omega_{nx},\Phi)\partial]_0 \\
&\qquad - \phi_n\omega^\dagger_{nt_A}\Phi - [M(A)\phi_n\partial^{-1}\omega_n^\dagger\partial\}]_0\Phi \Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_n \Big(-\phi_n\omega_n^\dagger \Phi_{t_A} + \phi_n \Omega(\omega_{nx},\Phi)_{t_A} + [M(A)\phi_n]_0 \Omega(\omega_{nx},\Phi) \\
&\qquad -\phi_n \omega_{nt_A}^\dagger \Phi - [M(A)\phi_n]_0 \omega_n^\dagger \Phi \Big) \\
&= \sum_n \Big(\phi_n (\Omega(\omega_{nx},\Phi) - \omega_n^\dagger \Phi) \Big)_{t_A} \;=\; (\Phi_z)_{t_A} \quad .
\end{aligned}
$$

For $(*)$ one omits the projection ≥ 0 in the first term, as the negative orders do not contribute. For the adjoint eigenfunctions we note

$$
\begin{aligned}
\omega_{t_A} &= -[\partial^{-1} M(A)^* \partial \omega]_0 \;=\; -\,res(\partial^{-1} M(A)^* \partial \omega \partial^{-1}) \\
&= -res(\partial^{-1} M(A)^* \omega_x \partial^{-1}) \;=\; res(\partial^{-1} \omega_x^\dagger M(A) \partial^{-1})^\dagger \quad ,
\end{aligned}
$$

so that one obtains

$$
\begin{aligned}
(\omega_{t_A}^\dagger)_z &= \sum_n \Big(\; res(\partial^{-1}(\Omega(\omega_x,\phi_n)\omega_n^\dagger)_x M(A)\partial^{-1}) - res(\partial^{-1}\omega_x^\dagger[\phi_n\partial^{-1}\omega_n^\dagger\partial, M(A)]_{\geq 1}\partial^{-1}) \\
&\overset{(*)}{=} \sum_n \Big(\; res(\Omega(\omega_x,\phi_n)\omega_n^\dagger M(A)\partial^{-1}) - res(\partial^{-1}\Omega(\omega_x,\phi_n)\omega_n^\dagger \partial M(A)\partial^{-1}) \\
&\qquad -res(\partial^{-1}\omega^\dagger\phi_n\partial^{-1}\omega_n^\dagger\partial M(A)\partial^{-1}) + res(\partial^{-1}\omega_x^\dagger M(A)\phi_n\partial^{-1}\omega_n^\dagger) \Big) \\
&\overset{(**)}{=} \sum_n \Big(-\,res(\partial^{-1}\Omega(\omega_x,\phi_n)\omega_n^\dagger\partial M(A)\partial^{-1}) \\
&\qquad -res(\partial^{-1}\{\partial\Omega(\omega_x,\phi_n) - \Omega(\omega_x,\phi_n)\partial\}\partial^{-1}\omega_n^\dagger\partial M(A)\partial^{-1}) \\
&\qquad +res(\partial^{-1}\omega_x^\dagger M(A)\phi_n\partial^{-1})\omega_n^\dagger \Big) \\
&= \sum_n \Big(-\,\Omega(\omega_x,\phi_n) res(\partial^{-1}\omega_n^\dagger\partial M(A)\partial^{-1}) + res(\partial^{-1}\omega_x^\dagger M(A)\phi_n\partial^{-1})\omega_n^\dagger \Big) \\
&= \sum_n \Big(\; \Omega(\omega_x,\phi_n) res(\partial^{-1}\omega_{nx}^\dagger M(A)\partial^{-1}) + res(\partial^{-1}\omega_x^\dagger M(A)\phi_n\partial^{-1})\omega_n^\dagger \Big) \\
&= \sum_n \Big(\Omega(\omega_x,\phi_n)\omega_{nt_A}^\dagger + \Omega(\omega_x,\phi_n)_{t_A}\omega_n^\dagger \Big) \;=\; \sum_n \Big(\Omega(\omega_x,\phi_n)\omega_n^\dagger \Big)_{t_A} \;=\; (\omega_z^\dagger)_{t_A} \quad .
\end{aligned}
$$

For $(*)$ one uses that in the second term $[\sum_n \phi_n\partial^{-1}\omega_n^\dagger\partial, M(A)]_{<1}$ does not contribute to the residue. For $(**)$ one observes that the first term vanishes, as $M(A)\partial^{-1}$ is a differential operator.

Proof of Theorem 6:

One derives the identity

$$
\begin{aligned}
(W^*B\partial W)_{t_A} &= -W^*\Big([WAW^{-1}\partial^{-1}]_{<0}\partial\Big)^* B\partial W - W^*B\partial\,[WAW^{-1}]_{<1}W \\
&= -W^*\partial\Big([\partial^{-1}W^{-1*}A^*W^*]_{<0}B\partial + B[WAW^{-1}]_{<1}\Big)W \\
&= -W^*\partial\Big([\partial^{-1}W^{-1*}A^*W^*B\partial]_{<1} + [BWAW^{-1}]_{<1}\Big)W \\
&= -W^*\partial\Big([\partial^{-1}W^{-1*}A^*C\partial W^{-1}]_{<1} + [\partial^{-1}W^{-1*}C\partial AW^{-1}]_{<1}\Big)W \\
&= -W^*\partial\,[\partial^{-1}W^{-1*}\;(A^*C\partial + C\partial A)\;W^{-1}]_{<1}W \quad ,
\end{aligned}
$$

so that the modified KP flows subject to (7.13) are tangent to the constraint (7.12). For the differential operators one finds

$$
\begin{aligned}
M(A)^*B\partial + B\partial M(A) &= \Big([WAW^{-1}\partial^{-1}]_{\geq 0}\partial\Big)^* B\partial + B\partial\,[WAW^{-1}]_{\geq 1} \\
&= \partial\Big([\partial^{-1}W^{-1*}A^*W^*]_{\geq 0}B\partial + B[WAW^{-1}]_{\geq 1}\Big) \\
&= \partial\Big([\partial^{-1}W^{-1*}A^*W^*B\partial]_{\geq 1} + [BWAW^{-1}]_{\geq 1}\Big) \\
&= \partial\Big([\partial^{-1}W^{-1*}A^*C\partial W^{-1}]_{\geq 1} + [\partial^{-1}W^{-1*}C\partial AW^{-1}]_{\geq 1}\Big) \\
&= \partial\,[\partial^{-1}W^{-1*}\,\Big(A^*C\partial + C\partial A\Big)W^{-1}]_{\geq 1} \quad .
\end{aligned}
$$