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A note on the Poisson brackets associated with Lax operators

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Abstract

Modifications of the standard Poisson brackets associated with differential scattering operators are considered. A linear bracket originates from a non-standard r-matrix on the algebra of pseudo-differential operators. Two quadratic brackets are investigated which provide Hamiltonian formulations for various reductions of the modified KP hierarchy.

Keywords: Integrable Systems, Solitons, KP Hierarchy, Poisson Brackets, Hamiltonian Structures, r-Matrix

1 Introduction

In their fundamental papers [1] Gelfand and Dikii considered the Hamiltonian structures of the soliton equations associated with (scalar) Lax operators of the type

$$L = \partial^N + u_{N-2}\partial^{N-2} + u_{N-1}\partial^{N-1} \ldots + u_0 \quad . \tag{1.1}$$

The corresponding hierarchies of generalized KdV equations for the fields $u_0, \ldots, u_{N-2}$ are encoded in the Lax equations

$$L_{t_q} = [(L^q)_{\geq 0}, L] \quad , \tag{1.2}$$

where the subscript ≥ 0 denotes the projection of the pseudo-differential fractional powers L^q onto their non-negative differential orders. In particular, the case $N = 2$ represents the KdV hierarchy proper, whereas $N = 3$ yields the Boussinesq hierarchy. The bi-Hamiltonian formulations for these equations arise from two abstract Poisson brackets associated with the Lax operator which are given by the following compatible Poisson tensors

$$\begin{aligned} P_1(L): \quad \nabla H \quad &\to \quad [L, (\nabla H)_{<0}]_{\geq 0} \quad , \\ P_2(L): \quad \nabla H \quad &\to \quad (L\, \nabla H)_{\geq 0} L - L\, (\nabla H\, L)_{\geq 0} \quad . \end{aligned} \tag{1.3}$$

The linear tensor is the Lie Poisson structure of the Adler-Kostant-Symes scheme [2], whereas for the quadratic tensor a Lie algebraic interpretation in terms of classical r-matrices was given by Semenov [3]. In general the compatible Poisson tensors

$$\begin{aligned} P_1(L): \quad \nabla H &\rightarrow [r(\nabla H), L] + r^*([\nabla H, L]) \quad , \\ P_2(L): \quad \nabla H &\rightarrow [r(L\,\nabla H + \nabla H\,L), L] + Lr^*([\nabla H, L]) + r^*([\nabla H, L])L \end{aligned} \tag{1.4}$$

can be associated with Lax operators. Here L may be an element of an arbitrary associative algebra g endowed with a bi-invariant metric given by a trace form. The corresponding linear bracket is a Poisson bracket for arbitrary r-matrices, whereas for the second bracket additional technical conditions on r are required. In particular, both r and its skew-symmetric part have to satisfy the (modified) Yang-Baxter equation (for details see Refs. [4, 5], where the usual unitarity condition for r was generalized and a further cubic bracket was given). A generalization of the quadratic bracket was recently proposed by Suris [6]. We will use his construction to prove the results of Theorem 3 in the following section.

Up to irrelevant constants, the tensors (1.3) arise as a particular example, when the algebra of pseudo-differential symbols

$$g = g_{\geq 0} \oplus g_{<0} = \{\sum_{n\geq 0} u_n \partial^n\} \oplus \{\sum_{n<0} \partial^n u_n\} \tag{1.5}$$

with the r-matrix $r(A) = A_{\geq 0} - A_{<0}$ is considered. Due to the unitarity (skew-symmetry) of r with respect to Adler's trace duality [2] both tensors (1.4) generate Poisson brackets on g. For the Casimir elements $H = tr(L^q)/q$ the associated Hamiltonian vector fields are the Lax equations (1.2), which give rise to the KP hierarchy for a general pseudo-differential $L \in g$. The restriction to Lax operators of the form (1.1) represents a reduction to the integrable equations of the Gelfand-Dikii hierarchy in 1+1 dimensions for which the bi-Hamiltonian formulations are given by (1.3)/(1.4). Indeed, the linear Poisson tensor admits a proper restriction to operators of the form (1.1), where (1.4.P_1) yields (1.3.P_1). The quadratic tensor will in general produce a term of order ∂^{N-1}, so that a term $u_{N-1}\partial^{N-1}$ has to be included in (1.1). Then (1.4.P_2) leads to (1.3.P_2). Dirac reduction has to be used to generate a quadratic bracket preserving the constraint $u_{N-1} = 0$. The reduced structure is given by the Poisson tensor

$$P_2^{(red)}(L) : \nabla H \quad \rightarrow \quad (L\,\nabla H)_{\geq 0} L - L(\nabla H\,L)_{\geq 0} + \frac{1}{N}\,[\,D^{-1}(res([\nabla H, L]))\,,\;L\,] \quad , \tag{1.6}$$

where res represents the usual residue (i.e. the coefficient of the differential order ∂^{-1}) and $D^{-1} = \int^x$ is an integration. The reduced bracket is again local, as the residue of a commutator is in the image of the differential operator [2]. It provides the "second" Hamiltonian operator of the Gelfand-Dikii hierarchies.

Recently there has been much interest in reductions of the KP hierarchy via the symmetry generated by squared eigenfunctions (e.g. [7-12]). It was found that the resulting constrained KP flows (1.2) are characterized by pseudo-differential Lax operators of the form

$$L = \partial^N + u_{N-2}\partial^{N-2} + \ldots + u_0 + \sum_{m=1}^{M} \phi_m \partial^{-1} \psi_m \quad , \tag{1.7}$$

which turns out to be an invariant operator manifold for the KP dynamics. The evolution (1.2) implies that ϕ_m and ψ_m are eigenfunctions and adjoint eigenfunctions of the KP hierarchy, as they satisfy the evolution equations

$$\phi_{mt_q} = (L^q)_{\geq 0}\phi_m \quad , \quad \psi_{mt_q} = -(L^q)^*_{\geq 0}\psi_m \quad . \tag{1.8}$$

These, however, do not represent linear equations, as the fields ϕ_m, ψ_m enter nonlinearly via L^q.

The Hamiltonian formulations for these systems are again given by (1.4) with the "standard" r-matrix $r(A) = A_{\geq 0} - A_{<0}$. In particular, it was shown in Ref. [13] that these tensors admit a restriction/reduction to the operator manifold (1.7).

In the following we will establish the analogous results for constraint flows of the *modified* KP hierarchy.

2 Results for constrained modified KP flows

The modified KP hierarchy arises when one considers a different decomposition $g = g_{\geq 1} \oplus g_{<1}$ of the underlying algebra. The subscript ≥ 1 represents the projection to differential orders larger or equal to 1. Hence, in contrast to the original construction zero order terms are now regarded as elements of the "negative" algebra. In general, we will use the notation

$$L_j = \begin{cases} v_j \partial^j & \text{for } j \geq 0 \ , \\ \partial^j v_j & \text{for } j < 0 \end{cases} \quad , \quad L_{\geq j} = \sum_{n \geq j} L_n \quad , \quad L_{<j} = \sum_{n<j} L_n \quad . \tag{2.1}$$

for $L = \sum_{n \geq 0} v_n \partial^n + \sum_{n<0} \partial^n v_n$.

The relevance of this "non-standard" decomposition in the context of soliton systems had already been observed by Reiman [14]. The resulting linear map $\tilde{r} : A \to A_{\geq 1} - A_{<1}$ is again an r-matrix, as both $g_{\geq 1}$ as well as $g_{<1}$ form sub-(Lie)algebras of g. The associated Lax equations

$$L_{t_q} = [(L^q)_{\geq 1}, L] \tag{2.2}$$

result in the modified KP hierarchy for generic pseudo-differential L [15-17]. One may consider reductions to 1+1 dimensional hierarchies by restricting the Lax operator to the form

$$L = \partial^N + v_{N-1}\partial^{N-1} + v_{N-2}\partial^{N-2} \ldots + v_1\partial \tag{2.3}$$

or

$$L = \partial^N + v_{N-1}\partial^{N-1} + v_{N-2}\partial^{N-2} \ldots + v_1\partial + v_0 + \partial^{-1}\Psi \tag{2.4}$$

or

$$L = \partial^N + v_{N-1}\partial^{N-1} + v_{N-2}\partial^{N-2} \ldots + v_1\partial + v_0 + \partial^{-1}\Psi + \sum_{m=1}^{m} \phi_m \partial^{-1}\psi_m \quad . \tag{2.5}$$

These "modified" equations have been studied extensively in Refs. [15-19], where also a further class of integrable Dym-type equations associated with a decomposition $g = g_{\geq 2} \oplus g_{<2}$ is considered.

Operators of the form (2.3) arise from gauge transformations $\tilde{L} = \Phi^{-1} \circ L \circ \Phi$, where L is given by (1.1). If Φ is an eigenfunction satisfying $L\Phi = 0$, then $\tilde{L}$ is of the form (2.3). Further, if Φ satisfies the time evolution $\Phi_{t_q} = (L^q)_{\geq 0}\Phi$, then the dynamics (1.2) yields (2.2) for $\tilde{L}$ (see e.g. Ref. [17]). Similarly, operators of the form (2.4)/(2.5) arise from gauge transformations of operators (1.7)[13].

The linear tensor (1.4) with $\tilde{r}$ gives rise to the Poisson tensor

$$\tilde{P}_1(L) = [\nabla H, L]_{<-1} - [(\nabla H)_{<1}, L] = [(\nabla H)_{\geq 1}, L]_{<-1} - [(\nabla H)_{<1}, L]_{\geq -1} \quad , \tag{2.6}$$

which reduces to $\tilde{P}_1(L) = [L, (\nabla H)_{<1}]_{\geq -1}$ for $L \in g_{\geq -1} = g^*_{<1}$, i.e. for operators of the form (2.4). The tensor (2.6) provides a Hamiltonian formulation for the flows (2.2) with the Hamiltonians $H = tr(L^{q+1})/(q+1)$, i.e. $\nabla H = L^q$. The restriction of the dynamics to operators of the form (2.4) and (2.5), respectively, originates from the Hamiltonian nature, as (2.4) and (2.5) constitute integral manifolds of the linear bracket. However, no proper Hamiltonian restriction to operators of the form (2.3) is possible.

Theorem 1:

The linear Poisson tensor (2.6) admits a proper restriction to Lax operators of the form (2.4) and (2.5).

For the proof one verifies that the image of the Poisson tensor is tangent to the manifold of operators given by (2.5). One calculates the associated Hamiltonian system as

$$\begin{aligned}
\dot{L}_{\geq 0} &= [L_{\geq 1}, (\nabla H)_{<1}]_{\geq 0} \quad , \\
\dot{\Psi} &= -res(\partial^{-1}\Psi(\nabla H)_{\geq 1}) - res([\nabla H, L]) \quad \equiv \quad -((\nabla H)_{\geq 1})^* \psi - res([\nabla H, L]) \\
\dot{\phi}_m &= (\nabla H)_{\geq 1}\phi_m \\
\dot{\psi}_m &= -res(\partial^{-1}\psi_m(\nabla H)_{\geq 1}) \quad \equiv \quad -((\nabla H)_{\geq 1})^* \psi_m \quad .
\end{aligned} \tag{2.7}$$

With

$$\nabla H = (\nabla H)_{\geq 1} + \frac{\delta H}{\delta \Psi} + \partial^{-1}\frac{\delta H}{\delta v_0} + \ldots + \partial^{-N}\frac{\delta H}{\delta v_{N-1}} \tag{2.8}$$

it is a straightforward exercise to compute the matrix differential operator which represents the Poisson tensor in the coordinates $v_{N-1}, \ldots, v_0, \Psi, \phi_m, \psi_m$. Here the elements dual to $\sum_m \phi_m \partial^{-1}\psi_m$ are represented by a differential operator $(\nabla H)_{\geq 1}$ satisfying

$$(\nabla H)_{\geq 1}\phi_m = \frac{\delta H}{\delta \psi_m} - \frac{\delta H}{\delta \Psi}\phi_m \quad , \quad ((\nabla H)_{\geq 1})^* \psi_m = \frac{\delta H}{\delta \phi_m} - \frac{\delta H}{\delta \Psi}\psi_m \quad , \tag{2.9}$$

so that with (2.8) and (2.9) Adler's trace duality yields the usual duality bracket

$$tr(\nabla H\ \dot{L}) \ = \ \int res(\nabla H\ \dot{L})\ dx \ = \ \int \Big\{ \sum_{n=0}^{N} \frac{\delta H}{\delta v_n}\dot{v}_n + \frac{\delta H}{\delta \Psi}\dot{\Psi} + \sum_{m=1}^{M} \big(\frac{\delta H}{\delta \phi_m}\dot{\phi}_m + \frac{\delta H}{\delta \psi_m}\dot{\psi}_m \big) \Big\}\ dx \tag{2.10}$$

of variational calculus. The resulting Hamiltonian equation is

$$\begin{aligned}
\dot{L}_{\geq 1} &= [L_{\geq 1}, (\nabla H)_{<1}]_{\geq 1} \ , \\
\dot{v}_0 &= [L_{\geq 1}, (\nabla H)_{<-1}]_0 + \Big\{ (L_{\geq 1}\partial^{-1}) - (L_{\geq 1}\partial^{-1})^* \Big\} \frac{\delta H}{\delta v_0} + L_{\geq 1}\frac{\delta H}{\delta \Psi} \ , \\
\dot{\Psi} &= res([L_{\geq 1}, (\nabla H)_{<-1}] - (L_{\geq 1})^* \frac{\delta H}{\delta v_0} + \phi_m \frac{\delta H}{\delta \phi_m} - \psi_m \frac{\delta H}{\delta \psi_m} \ , \\
\dot{\phi}_m &= \ \ \frac{\delta H}{\delta \psi_m} - \phi_m \frac{\delta H}{\delta \Psi} \ , \\
\dot{\psi}_m &= -\frac{\delta H}{\delta \phi_m} + \psi_m \frac{\delta H}{\delta \Psi} \ .
\end{aligned} \tag{2.11}$$

Further, the dynamics of $L_{\geq 1}$ involves no term of order N, so that one finds a proper restriction to operators of the form (2.5). With $\phi_m = \psi_m = 0$ the Poisson matrix decouples into separate diagonal blocks, so that the block associated with the fields v_i and Ψ provides a proper Poisson structure for operators of the form (2.4).

With

$$v_0 = \Phi + \omega + \sum_m \phi_m \omega_m \ , \ \ \Psi = -\omega_x \ \ , \ \ \psi_m = -\omega_{mx} \tag{2.12}$$

one might reparametrize (2.5) as

$$L = \partial^N + v_{N-1}\partial^{N-1} + \ldots + v_1\partial \ + \ \Phi + \partial^{-1}\omega\partial \ + \ \sum_{m=1}^{M} \phi_m \partial^{-1}\omega_m\ \partial \ \ . \tag{2.13}$$

In these coordinates the Hamiltonian system (2.7) obtains the form

$$\begin{aligned}
\dot{L}_{\geq 1} &= [L_{\geq 1}, (\nabla H)_{<1}]_{\geq 1} \ , \\
\dot{\Phi} &= (\nabla H)_{\geq 1}\Phi - [\nabla H, L]_0 - D^{-1}(res([\nabla H, L]) \ , \\
\dot{\phi}_m &= (\nabla H)_{\geq 1}\phi_m \ , \\
\dot{\omega} &= res(\partial^{-1}\omega_x(\nabla H)_{\geq 1}\partial^{-1}) + D^{-1}(res([\nabla H, L])) \ , \\
\dot{\omega}_m &= res(\partial^{-1}\omega_{mx}(\nabla H)_{\geq 1}\partial^{-1}) \ .
\end{aligned} \tag{2.14}$$

The Poisson tensor is non-local in these coordinates, as integrations D^{-1} arise from a corresponding parametrization of ∇H. Still, the form (2.13) seems more natural, when the Casimir elements $tr(L^{q+1})/(q+1)$ are considered as Hamiltonians. The soliton hierarchies associated with this construction are given by

$$\begin{array}{rclcrcl} (L_{\geq 1})_{t_q} & = & [(L^q)_{\geq 1}, L]_{\geq 1} \ , & & & & \\ \Phi_{t_q} & = & (L^q)_{\geq 1}\Phi & , & \omega_{t_q} & = & -(\partial^{-1}((L^q)_{\geq 1})^*\partial)\omega \ , \\ \phi_{m\,t_q} & = & (L^q)_{\geq 1}\phi_m & , & \omega_{m\,t_q} & = & -(\partial^{-1}((L^q)_{\geq 1})^*\partial)\omega_m \ . \end{array} \tag{2.15}$$

Hence, Φ and ϕ_m satisfy the evolution equations for eigenfunctions, whereas ω and ω_m are integrated adjoint eigenfunctions. We note that due to the absence of the zero order term the operators $\partial^{-1}((L^q)_{\geq 1})^*\partial$ are still differential operators. We remark that the equations (2.15) admit the trivial solutions $\phi_m = const$, $\omega_m = const$, so that the Φ and ω terms in (2.13) may be regarded as part of the following sum with "squared eigenfunction" pairs, in which the corresponding partner is chosen to be identically 1.

With these considerations a first Hamiltonian structure for the equations (2.2)/(2.15) has been established via the linear tensor (2.6). The natural question arises whether there is a second Poisson bracket giving rise to a bi-Hamiltonian structure.

As discussed in Ref. [5], Semenov's quadratic tensor (1.4) does *not* provide a Poisson bracket for the modified r-matrix $\tilde{r}$, as it is neither skew-symmetric nor does its skew-symmetric part satisfy the Yang-Baxter equation. Yet, it was pointed out in Ref. [19] that this tensor is still capable of producing suitable Poisson structures via Dirac reduction. In Ref. [13] it was shown that some terms have to be added to obtain a proper quadratic bracket on g:

Theorem 2:

The tensor

$$\begin{array}{rcl} \tilde{\mathcal{P}}_2(L) \ : \ \nabla H & \to & (L\,\nabla H)_{\geq 1}L - L\,(\nabla H\,L)_{\geq 0} + L\,(L\,\nabla H)_0 \\ & & -\partial^{-1}res([\nabla H, L])\,L + [D^{-1}(res([\nabla H, L]), L] \end{array} \tag{2.16}$$

defines a Poisson bracket on g which is compatible with the linear tensor (2.6). It admits a proper restriction to Lax operators of the form (2.4) and (2.5). For the Hamiltonians $H = tr(L^q)/q$ the associated dynamics is given by (2.2).

The Poisson properties of (2.16) and the restrictions to (2.4) were derived in Ref. [13]. For the restriction to operators of the more general form (2.5) one decomposes $\dot{L} = \tilde{\mathcal{P}}_2(L)\nabla H$ into its differential orders and a straightforward calculation yields the consistent Hamiltonian equation

$$\begin{array}{rcl} \dot{\Psi} & = & -((\nabla H\,L)_{\geq 1})^*\Psi - \Psi[\nabla H, L]_0 - (L_{\geq 0})^*\rho + \psi_m D^{-1}(\phi_m\rho) \ , \\ \dot{\phi}_m & = & (L\,\nabla H)_{\geq 1}\phi_m + \phi_m D^{-1}(\rho) - D^{-1}(\phi_m\rho) \ , \\ \dot{\psi}_m & = & -((\nabla H\,L)_{\geq 1})^*\psi_m - \psi_m[\nabla H, L]_0 - \psi_m D^{-1}(\rho) \ , \end{array} \tag{2.17}$$

where $\rho = res([\nabla H, L])$. The dynamics of $L_{\geq 0}$ is directly given as the corresponding projection of (2.16). Thus, the reduction properties are shown. Evidently, the restriction to Lax operators

(2.4) is possible by choosing $\phi_m = \psi_m = 0$.

No proper restriction to (2.3) is possible. Hence, Dirac reduction has to be invoked to obtain the quadratic bracket associated with the constraint $v_0 = \Psi = 0$. Surprisingly, this reduction process can be carried out formally and yields a *local* Poisson structure which is naturally extended to the whole of g:

Theorem 3:

The tensor

$$\begin{aligned} \hat{\mathcal{P}}_2(L) \ : \ \nabla H \ \ \to \ \ & (L\,\nabla H)_{\geq 0} L - L\,(\nabla H\,L)_{\geq 1} - (\nabla H\,L)_0\,L \\ & -L\,\partial^{-1} res([\nabla H, L]) - [D^{-1}(res([\nabla H, L]), L] \end{aligned} \tag{2.18}$$

defines a Poisson bracket on g which is compatible with the linear tensor (2.6). It admits a proper restriction to Lax operators of the form (2.3), where it coincides with the Dirac reduction of (2.16). For the Hamiltonians $H = tr(L^q)/q$ the associated dynamics is given by (2.2).

Both the Poisson property and the reduction result are checked by a straightforward, yet lengthy and cumbersome computation. The compatibility with (2.6) follows from the usual deformation argument that $L \to L + \epsilon 1$ produces the linear tensor as the term linear in ϵ.

We finally remark that the tensors (2.16) and (2.18) may be (formally) derived from the general formulation of a quadratic bracket recently suggested by Suris [6]. There it was found that on any algebra g equipped with a non-degenerate invariant metric $<a, bc>=<ab, c>$ the quadratic tensor

$$\mathcal{P}'(L) \ : \ \ \nabla H \ \ \to A_1(L\,\nabla H)\,L - LA_2(\nabla H\,L) + S(\nabla H\,L)L - LS^*(L\,\nabla H) \tag{2.19}$$

defines a Poisson structure on $g = g^*$, if the linear maps $A_{1,2} : g \to g$ are skew-symmetric solutions of the modified Yang-Baxter equation

$$[A(\xi), A(\eta)] + [\xi, \eta] = A(\,[A(\xi), \eta] + [\xi, A(\eta)]\,) \tag{2.20}$$

and the linear map $S : g \to g$ with adjoint S^* satisfies

$$\begin{aligned} S\,(\,[A_2(\xi), \eta] + [\xi, A_2(\eta)]\,) \ &= \ [\,S(\xi), S(\eta)\,] \ , \\ S^*(\,[A_1(\xi), \eta] + [\xi, A_1(\eta)]\,) \ &= \ [S^*(\xi), S^*(\eta)] \end{aligned} \tag{2.21}$$

for all $\xi, \eta \in g$. It was shown in Ref. [6] that a unified description of the Hamiltonian structures of the Toda model as well as the relativistic Toda model is provided by this construction.

It can be verified that on the algebra g given by (1.5) the maps

$$\begin{aligned} A_1 \ &: \ \xi \ \to \ \xi_{\geq 1} - \xi_0 + \xi_{-1} - \xi_{<-1} - 2\,D^{-1}(res(\xi)) \ , \\ A_2 \ &: \ \xi \ \to \ \xi_{\geq 0} - \xi_{<0} + 2\,D^{-1}(res(\xi)) \ , \\ S \ &: \ \xi \ \to \ -2\,\xi_{-1} + 2\,D^{-1}(res(\xi)) \ , \\ S^* \ &: \ \xi \ \to \ -2\,\xi_0 \ - 2\,D^{-1}(res(\xi)) \ , \end{aligned} \tag{2.22}$$

formally satisfy these conditions. With these choices the tensor (2.19) results in (2.16). Swapping A_1 and A_2 and simultaneously replacing S by S^* the tensor (2.19) results in (2.18). Thus, both tensors (2.16) and (2.18) arise from the same algebraic structures, where the exchange $A_1 \leftrightarrow A_2$, $S \leftrightarrow S^*$ corresponds to the re-definition of the algebraic multiplication $\xi \circ \eta := \eta\xi$ instead of $\xi \circ \eta := \xi\eta$.

We note that the integration D^{-1} is not well-defined in (2.22), so that the argument remains formal. However, in (2.16) as well as in (2.18) this problem does not arise, as the residues of the commutators can be properly integrated. We further remark that with the above choices one finds $[S(\xi), S(\eta)] = [S^*(\xi), S^*(\eta)] = 0$, so that the conditions (2.21) are still satisfied if one replaces S by $-S$. However, the resulting bracket is intrinsically non-local and does not seem to be interesting.

Example 1: According to Theorems 1 and 2 the bi-Hamiltonian structure of the soliton hierarchy associated with the Lax operator $L = \partial + v + \partial^{-1}\Psi$ can be obtained from (2.6) and (2.16). Indeed, with

$$\nabla H = \frac{\delta H}{\delta \Psi} + \partial^{-1}\frac{\delta H}{\delta v} \tag{2.23}$$

one derives the Hamiltonian operators

$$\tilde{\mathcal{P}}_1(L) \ : \ \begin{pmatrix} \frac{\delta H}{\delta v} \\ \frac{\delta H}{\delta \Psi} \end{pmatrix} \ \longrightarrow \ \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta v} \\ \frac{\delta H}{\delta \Psi} \end{pmatrix} \tag{2.24}$$

and

$$\tilde{\mathcal{P}}_2(L) \ : \ \begin{pmatrix} \frac{\delta H}{\delta v} \\ \frac{\delta H}{\delta \Psi} \end{pmatrix} \ \longrightarrow \ \begin{pmatrix} 2D & , & D^2 + Dv \\ -D^2 + vD & , & D\Psi + \Psi D \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta v} \\ \frac{\delta H}{\delta \Psi} \end{pmatrix} \tag{2.25}$$

for the Kaup-Broer system

$$\begin{aligned} v_t &= v_{xx} + 2\,\Psi_x + 2\,v v_x \ , \\ \Psi_t &= -\Psi_{xx} + 2\,(v\Psi)_x \ , \end{aligned} \tag{2.26}$$

which arises from $L_t = [(L^2)_{\geq 1}, L]$ (see e.g. Refs. [18, 19]). Here $D = \partial/\partial x$ is the differential operator.

Example 2: The Lax operator $L = \partial^2 + 2v\partial$ gives rise to the modified KdV equation

$$4\,v_t = v_{xxx} - 6\,v v_x \tag{2.27}$$

via $L_t = [(L^{3/2})_{\geq 1}, L]$. With $\nabla H = \frac{1}{2}\partial^{-2}\delta H/\delta v$ the quadratic tensor of Theorem 3 produces the well-known Hamiltonian operator $\hat{\mathcal{P}}_2(L) \ : \ \delta H/\delta v \ \rightarrow \ -(\delta H/\delta v)_x$.

Example 3: The modified Boussinesq system [19]

$$\begin{aligned} 3\,v_{1t} &= -2v_{2xxx} + 3v_{1xx} + 2v_2 v_{1x} - 2v_{2x}v_1 - 2v_2 v_{2xx} \ , \\ 3\,v_{2t} &= -3v_{2xx} + 6v_{1x} - 2v_2 v_{2x} \end{aligned} \tag{2.28}$$

is associated with the Lax operator $L=\partial^3+v_2\partial^2+v_1\partial$ via $L_t=[(L^{2/3})_{\geq 1},L]$. The quadratic tensor of Theorem 3 yields the Hamiltonian operator

$$
\begin{aligned}
\hat{\mathcal{P}}_2(L)\ :\ & \begin{pmatrix} \frac{\delta H}{\delta v_1} \\ \frac{\delta H}{\delta v_2} \end{pmatrix} \\
\longrightarrow\ & \begin{pmatrix} 2D^3+v_1D+Dv_1+2v_2D^2-2D^2v_2-2v_2Dv_2 & , & -3(D+v_2)D \\ 3D(D-v_2) & , & -6D \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta v_1} \\ \frac{\delta H}{\delta v_2} \end{pmatrix} .
\end{aligned}
\tag{2.29}
$$

After the transformations $v_1=u+v_x/2+v^2/4,\ v_2=v$ and $u=a_x-a^2,\ v=b$ this operator diagonalizes and finally reduces to a constant operator:

$$
\begin{aligned}
\begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta v} \end{pmatrix} &\rightarrow \begin{pmatrix} \frac{1}{2}D^3+Du+uD & 0 \\ 0 & -6D \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta v} \end{pmatrix} , \\
\begin{pmatrix} \frac{\delta H}{\delta a} \\ \frac{\delta H}{\delta b} \end{pmatrix} &\rightarrow \begin{pmatrix} -\frac{1}{2}D & 0 \\ 0 & -6D \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta a} \\ \frac{\delta H}{\delta b} \end{pmatrix} .
\end{aligned}
\tag{2.30}
$$

The Hamiltonian for (2.28) is given by $H=\frac{3}{2}\,tr(L^{2/3})=\frac{1}{27}\int(2v_2^3-9v_1v_2)\,dx$.

3 Conclusions

The modified KP hierarchy encoded in $L_{t_q}=[(L^q)_{\geq 1},L]$ admits reductions to soliton hierarchies in 1+1 dimensions by requiring that the Lax operator has the special form (2.5). From the results of Ref. [20] if follows that squared eigenfunctions generate a symmetry transformation of the modified KP hierarchy via generalized Lax equations of the form

$$
L_z=-[\sum_{m=1}^{M}\phi_m\partial^{-1}\omega_m\partial\ ,\ L\ ]\ , \tag{3.31}
$$

where z is the evolution parameter and ϕ_m, ω_m are eigenfunctions and integrated adjoint eigenfunctions. Hence, with $L_{t_q}=[(L^q)_{\geq 1},L]=-[(L^q)_{<1},L]$ the requirement

$$
(L^q)_{<1}=\sum_m\phi_m\partial^{-1}\omega_m\partial \tag{3.32}
$$

may be understood as a symmetry reduction $L_{t_q}=L_z$ of the modified KP hierarchy. This implies that L^q is of the form (2.13), provided in two of the squared eigenfunction terms an (adjoint) eigenfunction is chosen to be identically 1 to match the terms involving Φ and ω. Thus, the Lax equations (2.2) with L replaced by the operator $\tilde{L}=L^q$ coincide with these constrained flows, where $\tilde{L}$ is of the form (2.5)/(2.13).

The (bi-)Hamiltonian formulation for these systems has been found. A linear bracket is provided by the r-matrix approach, whereas a quadratic Poisson structure is identified through an appropriate modification of Semenov's quadratic bracket. In particular, the Hamiltonian formulation for the modified Gelfand-Dikii hierarchies associated with scattering problems of the type (2.3) is established via Dirac reduction. The result may be reinterpreted as yet another quadratic bracket which is covered by Suris' generalization of Semenov's quadratic Poisson structure.

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