in: *Nonlinear Physics. Theory and Experiment* (Proceedings of the 1st Workshop on Nonlinear Physics, Gallipoli, 29.6.–7.7.1995), pp. 233–240, E. Alfinito, M. Boiti, L. Martina and F. Pempinelli (eds.), World Scientific, Singapore, 1996.

DARBOUX TRANSFORMATIONS FOR INTEGRABLE LATTICE SYSTEMS

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A framework for a general description of Darboux transformations for Lax representations of discrete integrable systems is presented. The Lax equations are regarded as dynamical systems in the algebra of shift operators which is embedded in an algebra of pseudo-difference symbols. Gauge transformations are given by operators satisfying a dressing equation in this space. Special dressing operators are found which are parametrized by (adjoint) eigenfunctions of the Lax system. They give rise to Darboux like transformations as well as adjoint and binary versions. Reductions to finite operators are discussed.

1 Introduction

From a dynamical point of view Bäcklund transformations [1] may be regarded as invariant relations ("submanifolds") between evolution equations. For an integrable system, given by a Lax representation $L_t = [M, L]$, they usually originate from certain invariances of the Lax operator. These are "similarity transformations" $L \to \tilde{L} = WLW^{-1}$ preserving the Lax equation, which have been termed "gauge transformations", "dressing transformations" or "Darboux transformations" (DTs) in various contexts. Darboux's original observation was [2] that a solution of the Schrödinger equation $L(u)\phi = \lambda\phi$, $L(u) = \partial^2 + u$, can be transformed into a solution $\tilde{\phi} = \phi_0(\phi_0^{-1}\phi)_x$ of another spectral problem $L(\tilde{u})\tilde{\phi} = \lambda\tilde{\phi}$ by means of an eigenfunction ϕ_0 satisfying $L(u)\phi_0 = \lambda_0\phi_0$:

$$\tilde{u} \;=\; u \;+\; 2\,(\ln(\phi_0))_{xx}\ . \tag{1}$$

Its relevance in the context of integrable equations is as follows. If one adjoins the time evolutions $\phi_t = M(u)\phi$, $M(u) = 4\partial^3 + 6u\partial + 3u_x$, for both ϕ and ϕ_0, then one finds $\tilde{\phi}_t = M(\tilde{u})\tilde{\phi}$ after some straightforward computation. By compatibility this implies that both the original potential u and the transformed potential $\tilde{u}$ satisfy the KdV equation $u_t = u_{xxx} + 6uu_x$. The map $u \to \tilde{u}$, triggered by the eigenfunction ϕ_0, may be regarded as the gauge transformation[a]

$$\begin{array}{rclcrclcl} L & = & L(u) & \to & \tilde{L} & = & L(\tilde{u}) & = & WL(u)W^{-1} \\ M & = & M(u) & \to & \tilde{M} & = & M(\tilde{u}) & = & WM(u)W^{-1} + W_tW^{-1} \end{array}$$

of the Lax operators with $W = \phi_0\, \partial \circ \phi_0^{-1}$ ($\circ$ indicates an operator product). Indeed, one computes

$$WL(u)W^{-1} \;=\; L(\tilde{u}) \;+\; \left(\frac{L(u)\phi_0}{\phi_0}\right)_x W^{-1}$$

and

$$WM(u)W^{-1} + W_tW^{-1} \;=\; M(\tilde{u}) - \left(\frac{\phi_{0t} - M(u)\phi_0}{\phi_0}\right)_x W^{-1} + 6\left(\frac{L(u)\phi_0}{\phi_0}\right)_x .$$

Hence, the spectral problem $L(u)\phi_0 = \lambda_0\phi_0$ is needed to preserve the form of L, and the time evolution $\phi_{0t} = M(u)\phi_0$ grants the invariance of the Lax equation.

The DT (1) is triggered by ϕ_0, a quantity satisfying *linear* equations. Elimination of ϕ_0 yields the (spatial part of the) usual auto-Bäcklund transformation $2(\tilde{u} + u) + (\int^x (\tilde{u} - u))^2 = 4\,\lambda_0$ of the KdV hierarchy.

Following lessons are to be learned from this well-known example:

i) DTs are "linearized" Bäcklund transformations, arising from a gauge (dressing) transformation of the Lax representation.

ii) The non-invertible dressing operator W needs to be inverted.

Although ii) looks absurd, this can be achieved in a simple technical manner. Embedding the differential operator W (with kernel spanned by ϕ_0) into the algebra of pseudo-differential symbols, $W^{-1} = \phi_0\, \partial^{-1} \circ \phi_0^{-1}$ is well-defined. However, W^{-1} may not be applied to L_2-functions, so that the transformation of L is not a similarity transformation in a proper sense and the spectrum of the Lax operator may be changed. Indeed, the DT adds λ_0 to the eigenvalues.

In the following a similar construction is used to define DTs for discrete Lax operators of the form

$$L = u_kT^k + u_{k-1}T^{k-1} + \cdots .$$

The coefficients $u_j = u_j(i)$ are sequences with possibly non-commuting components and $T : u(i) \to u(i+1)$ is the shift operator. The map $r : \sum_j u_jT^j \to$

[a] The Lax dynamics is preserved, automatically: $L = [M, L] \Rightarrow \tilde{L} = [\tilde{M}, \tilde{L}]$.

$\sum_{j\geq 0} u_j T^j - \sum_{j<0} u_j T^j$ provides a classical r-matrix [3] on the algebra of such operators, leading to the commuting hierarchy of integrable Lax equations [4]

$$L_{t_n} \;=\; [(L^n)_{\geq 0}, L] \;, \tag{2}$$

where $(.)_{\geq 0}$ denotes the projection onto non-negative powers of T. In the following we will derive DTs for these systems. It is easily verified that operators solving the commuting hierarchy of "dressing equations"

$$W_{t_n} \;=\; -\, W\,(L^n)_{\geq 0} \;+\; (W L^n W^{-1})_{\geq 0} W \tag{3}$$

generate transformations $L \to \tilde{L} = W L W^{-1}$ keeping the dynamics (2) invariant: $\tilde{L}_{t_n} = [(\tilde{L}^n)_{\geq 0}, \tilde{L}]$. Remarkably, the dressing equations (3) can be linearized, i.e., there are solutions parametrized by objects satisfying linear equations (eigenfunctions and adjoint eigenfunctions). These results are given in section 2.

We introduce the difference operator $\Delta = T - 1$. Formally, its inverse may be written as $\Delta^{-1} = T^{-1} + T^{-2} + \cdots$. However, it is more appropriate to interpret it as a pseudo-difference symbol [4] with the algebraic property

$$\Delta^{-1} \circ u \;=\; u^{(-1)} \Delta^{-1} - (\Delta u^{(-2)})\, \Delta^{-2} + (\Delta^2 u^{(-3)})\, \Delta^{-3} \mp \cdots \;,$$

where $(u^{(k)}(i) = (T^k u)(i) = u(i+k)$.

The evolution equations (2) are compatible with the reduction to Lax operators parametrized by finitely many fields

$$L \;=\; u_k T^k + u_{k-1} T^{k-1} + \cdots + u_m T^m + q\, T^m \Delta^{-1} \circ r^* \;, \tag{4}$$

where the functions q and r inherit the dynamics

$$q_{t_n} \;=\; (L^n)_{\geq 0}\, q \;, \quad r_{t_n} \;=\; -\,(L^n)^*_{\geq 0}\, r$$

of eigenfunctions and adjoint eigenfunctions, respectively. Here

$$(\, u_n T^n + u_{n-1} T^{n-1} + \cdots)^* \;=\; T^{-n} u_n^* + T^{1-n} u_{n-1}^* + \cdots$$

is the formal adjoint and u_j^* denotes transposition (u_j, q and r may be column vectors or matrices). Obviously the nonlocal term in (4) can be omitted, as the constraint $q = r = 0$ is compatible with the dynamics.

For the DTs $L \to W L W^{-1}$ of reduced Lax operators (4) it is not sufficient that W satisfies (3). This would only grant the invariance of the Lax equations (2), but the form of L may be changed by the transformation. Certain additional conditions will arise to preserve the form (4) (section 3).

We finally present a technical result which will be relevant for the transformations of the next sections. If ϕ and ψ satisfy the dynamics of eigenfunctions and adjoint eigenfunctions associated with an operator $M = m_0 + m_1 T + \cdots$, then there exists a potential integrating the "squared eigenfunction" $\psi^* \phi$:

Lemma: *If $\phi_t = M\phi$ and $\psi_t = -M^*\psi$, then there exists a potential $\Omega = \Omega(\psi^*, \phi)$ defined by the compatible equations*

$$\Omega^{(1)} - \Omega \; = \; \psi^* \phi \; , \quad \Omega_t \; = \; \mathrm{res}(\, \Delta^{-1} \circ \psi^* \circ M \circ \phi \circ \Delta^{-1} \,) \; .$$

Here $\mathrm{res}(\sum_j u_j T^j) = u_{-1}$ is the residue of a pseudo-difference operator. The compatibility of the two equations for Ω is easily checked. This potential is uniquely defined up to a constant, which does neither depend on the time nor the lattice site. It should be regarded as a bilinear function of ψ and ϕ.

2 Darboux transformations

In this section we look for solutions of the dressing hierarchy (3). It turns out that there are simple dressing operators parametrized by solutions of the linear equations associated with the Lax hierarchy. In the following the notion of "eigenfunctions" ϕ and "adjoint eigenfunctions" ψ of a Lax operator L refers to the dynamical equations

$$\phi_{t_n} = (L^n)_{\geq 0} \, \phi \; , \quad \psi_{t_n} = -(L^n)^*_{\geq 0} \, \psi \; . \tag{5}$$

No spectral equation such as $L\phi = \lambda\phi$ is assumed in this section. The following theorems are stated without proofs. They are the discrete analogues of the general description of (continuous) DTs as given in Ref. 5. The statements may be verified by direct computations, details are to be published elsewhere.

Theorem 1: *Let ϕ_0 and ψ_0 be (adjoint) eigenfunctions of L. Then the operators* [b]

$$\begin{array}{llll} a) & W \; = \; \phi_0^{(1)} \, \Delta \circ \phi_0^{-1} & & \textit{(DT)} \\ b) & W \; = \; (\psi_0^{*(-1)})^{-1} \, \Delta^{-1} \circ \psi_0^* & & \textit{(adjoint DT)} \\ c) & W \; = \; 1 \; - \; \phi_0 \, \Omega(\psi_0^*, \phi_0)^{-1} \, \Delta^{-1} \circ \psi_0^* & & \textit{(binary DT)} \end{array}$$

solve the hierarchy of dressing equations (3), so that the dynamics (2) is invariant under the transformation $L \to W L W^{-1}$.

[b] The inverse of the operator c) is $W^{-1} = 1 + \phi_0 \, \Delta^{-1} \circ (\Omega(\psi_0^*, \phi_0)^{(1)})^{-1} \psi_0^*$.

Darboux's original result for the Schrödinger operator featured a transformation mechanism to map eigenfunctions of L to eigenfunctions of $\tilde{L} = WLW^{-1}$. This, in principle, is achieved by $\phi \to \tilde{\phi} = W\phi$. However, due to the occurrence of the pseudo-difference symbol Δ^{-1} special care has to be taken to give a rigorous meaning to the application of W to a function. The heuristic principle is that $\Delta^{-1}\psi_0^*\phi$ has to be replaced by the squared eigenfunction potential $\Omega(\psi_0^*, \phi)$. The following theorem provides rigorous statements on the transformation of eigenfunctions, adjoint eigenfunctions and their bilinear potential:

Theorem 2: *Let ϕ_0 and ψ_0 be eigenfunctions and adjoint eigenfunctions, respectively, associated with L. Let W be one of the dressing operators of Theorem 1, generated by ϕ_0 and ψ_0. If ϕ is an eigenfunction of L, then*

$$\begin{array}{llll} a) & \tilde{\phi} = \phi^{(1)} - \phi_0^{(1)}\phi_0^{-1}\phi & & \textit{(DT)} \\ b) & \tilde{\phi} = (\psi_0^{*(-1)})^{-1}\,\Omega(\psi_0^*, \phi) & & \textit{(adjoint DT)} \\ c) & \tilde{\phi} = \phi - \phi_0\,\Omega(\psi_0^*, \phi_0)^{-1}\,\Omega(\psi_0^*, \phi) & & \textit{(binary DT)} \end{array}$$

is an eigenfunction of $\tilde{L} = WLW^{-1}$. An adjoint eigenfunction ψ of L is transformed into the adjoint eigenfunction $\tilde{\psi}$ of $\tilde{L}$ by

$$\begin{array}{llll} a) & \tilde{\psi}^* = (\Omega(\psi^*, \phi_0)\,\phi_0^{-1})^{(1)} & & \textit{(DT)} \\ b) & \tilde{\psi}^* = \psi^*\psi_0^{*-1}\psi_0^{*(-1)} - \psi^{*(-1)} & & \textit{(adjoint DT)} \\ c) & \tilde{\psi}^* = \psi^* - \Omega(\psi^*, \phi_0)^{(1)}\,(\Omega(\psi_0^*, \phi_0)^{(1)})^{-1}\psi_0^* & & \textit{(binary DT).} \end{array}$$

The squared eigenfunction potential of the new pair is given by

$$\begin{array}{llll} a) & \Omega(\tilde{\psi}^*, \tilde{\phi}) = \Omega(\psi^*, \phi_0)\phi_0^{-1}\phi - \Omega(\psi^*, \phi) & & \textit{(DT)} \\ b) & \Omega(\tilde{\psi}^*, \tilde{\phi}) = \psi^{*(-1)}(\psi_0^{*(-1)})^{-1}\,\Omega(\psi_0^*, \phi) - \Omega(\psi^*, \phi) & & \textit{(adjoint DT)} \\ c) & \Omega(\tilde{\psi}^*, \tilde{\phi}) = \Omega(\psi^*, \phi) - \Omega(\psi^*, \phi_0)\,\Omega(\psi_0^*, \phi_0)^{-1}\,\Omega(\psi_0^*, \phi) & & \textit{(binary DT).} \end{array}$$

We remark that the adjoint DT may be regarded as an inverse DT: from L_0 with eigenfunction ϕ_0 and trivial adjoint eigenfunction $\psi_0 = 0$ one may construct $L_1 = W_{10}L_0W_{10}^{-1}$ with $W_{10} = \phi_0\Delta \circ \phi_0^{-1}$. According to Theorem 2 there exists the (transposed) adjoint eigenfunction $\psi_1^* = (\phi_0^{(1)})^{-1}$, when putting $\Omega(\psi_0^*, \phi_0) = \Omega(0, \phi_0) = 1$. An adjoint DT $L_2 = W_{21}L_1W_{21}^{-1}$ with

$$W_{21} = (\psi_1^{*(-1)})^{-1}\,\Delta^{-1} \circ \psi_1^* = \phi_0\,\Delta^{-1} \circ (\phi_0^{(1)})^{-1} = W_{10}^{-1}$$

leads back to $L_2 = L_0$. Similarly, the binary DT may be understood as a composition of a DT and an adjoint DT:

$$
\begin{tikzcd}
& \boxed{L_0, \phi_0, \psi_0} \arrow[dl, "W_{10}\ (\mathrm{DT})"'] \arrow[dr, "W'_{10}\ (\text{adjoint DT})"] & \\
\boxed{L_1, \psi_1} \arrow[dr, "W_{21}\ (\text{adjoint DT})"'] & & \boxed{L'_1, \phi'_1} \arrow[dl, "W'_{21}\ (\mathrm{DT})"] \\
& \boxed{L_2} &
\end{tikzcd}
$$

Starting with the Lax operator L_0 and a pair of (adjoint) eigenfunctions ϕ_0, ψ_0, one may first use a DT $W_{10} = \phi_0^{(1)} \Delta \circ \phi_0^{-1}$ to obtain L_1 with the (transposed) adjoint eigenfunction $\psi_1^* = (\Omega(\psi_0^*, \phi_0)\phi_0^{-1})^{(1)}$, which is used to trigger an adjoint DT $W_{21} = (\psi_1^{*(-1)})^{-1} \Delta^{-1} \circ \psi_1^*$ to L_2. The diagram commutes: one may just as well start with the adjoint DT triggered by ψ_0 and use the image ϕ'_1 of ϕ_0 for a further DT. This results in the same operator L_2. The composed dressing operator $W_{21}W_{10} = W'_{21}W'_{10}$ mapping L_0 to L_2 defines the binary transformation.

Because of Theorem 2 DT's can be iterated, if a set of eigenfunctions $\phi_1, \ldots, \phi_n$ and/or adjoint eigenfunctions $\psi_1, \ldots, \psi_n$ is known. One may use ϕ_1, ψ_1 for a first transformation, then map the other (adjoint) eigenfunctions to new (adjoint) eigenfunctions and proceed from there. It is easily checked that the result of such an iteration does not depend on the ordering in which the triggering (adjoint) eigenfunctions are used. We present the composed dressing operators for iterated DTs and iterated binary DTs:

Theorem 3: *The iterated DT triggered by eigenfunctions $\phi_1, \ldots, \phi_n$ of L is given by a dressing operator $W = T^n + w_{n-1}T^{n-1} + \cdots + w_0$ satisfying (3). Its components are defined by the linear equations*

$$
(w_0, \ldots, w_{n-1}) \begin{pmatrix} \phi_1 & \ldots & \phi_n \\ \vdots & \ddots & \vdots \\ \phi_1^{(n-1)} & \ldots & \phi_n^{(n-1)} \end{pmatrix} = - \, (\phi_1^{(n)}, \ldots, \phi_n^{(n)}) \; .
$$

With the matrix $\Omega = (\Omega(\psi_i^*, \phi_j))$ the iterated binary DT triggered by $\phi_1, \ldots, \phi_n$

and the adjoint eigenfunctions $\psi_1, \ldots, \psi_n$ *is given by the dressing operator*

$$W \;=\; 1 \;-\; (\phi_1, \ldots \phi_n) \;\; \Omega^{-1} \begin{pmatrix} \Delta^{-1} \circ \psi_1^* \\ \vdots \\ \Delta^{-1} \circ \psi_n^* \end{pmatrix}$$

which satisfies (3).

These results are immediate consequences of the observation that the dressing operator of an iterated transformation has a kernel spanned by the triggering eigenfunctions. The effect of the iterated DTs on the coefficients of $L = u_k T^k + u_{k-1} T^{k-1} + \cdots$ is easily computed from $LW = W\tilde{L}$.

3 Darboux transformations for $L = u_k T^k + \cdots + u_m T^m + q\, T^m \Delta^{-1} \circ r^*$

So far only the dynamics (5) was imposed on the (adjoint) eigenfunctions generating the dressing transformations of Theorem 1. We now consider reductions of the hierarchy (2) to Lax operators of the form (4), where we have to ensure that the transformation to

$$\tilde{L} \;=\; W L W^{-1} \;=\; \tilde{u}_k T^k + \cdots + \tilde{u}_m T^m + \tilde{q}\, T^m \Delta^{-1} \circ \tilde{r}^* \tag{6}$$

keeps this form invariant. It turns out that spectral equations $L\phi_0 = \lambda \phi_0$, i.e.,

$$u_k \phi_0^{(k)} + u_{k-1} \phi_0^{(k-1)} + \;\cdots\; + u_m \phi_0^{(m)} + q\, \Omega(r^*, \phi_0)^{(m)} \;=\; \lambda\, \phi_0 \;, \tag{7}$$

and $(L^* \psi_0)^* = \mu \psi_0^*$, i.e.,

$$(\psi_0^* u_k)^{(-k)} + (\psi_0^* u_{k-1})^{(1-k)} + \;\cdots\; + (\psi_0^* u_m)^{(-m)} - \Omega(\psi_0^*, q)^{(1-m)}\, r^* \;=\; \mu\, \psi_0^* \;, \tag{8}$$

have to be imposed on the triggering (adjoint) eigenfunctions. The arbitrary constant in the square eigenfunction potential then can be chosen such that

$$\begin{aligned} \lambda\, \Omega(\psi_0^*, \phi_0)^{(-m)} \;-\; \mu\, \Omega(\psi_0^*, \phi_0) \;=\; & \operatorname{res}(T^{-m} \Delta^{-1} \circ \psi_0^* \circ L \circ \phi_0 \circ \Delta^{-1}) \\ & +\; \Omega(\psi_0^*, q)^{(-m)}\, \Omega(r^*, \phi) \end{aligned} \tag{9}$$

holds.

Theorem 4: *If* ϕ_0 *and* ψ_0 *satisfy the spectral equations (7) and (8), respectively, and* $\Omega(\psi_0^*, \phi_0)$ *is chosen according to (9), then the DTs of Theorem 1 yield* $\tilde{L} = W L W^{-1}$ *of the form (6) with*

$$\tilde{q} \;=\; \begin{cases} q^{(1)} - \phi_0^{(1)} \phi_0^{-1} q & \textit{(DT)} \\ (\psi_0^{*(-1)})^{-1}\, \Omega(\psi_0^*, q) & \textit{(adjoint DT)} \\ q - \phi_0 \Omega(\psi_0^*, \phi_0)^{-1}\, \Omega(\psi_0^*, q) & \textit{(binary DT)} \end{cases}$$

$$\text{and} \qquad \tilde{r}^* = \begin{cases} (\Omega(r^*,\phi_0)\,\phi_0^{-1})^{(1)} & \textit{(DT)} \\ r^*\psi_0^{*-1}\psi_0^{*(-1)} - r^{*(-1)} & \textit{(adjoint DT)} \\ r^* - \Omega(r^*,\phi_0)^{(1)}(\Omega(\psi_0^*,\phi_0)^{(1)})^{-1}\psi_0^* & \textit{(binary DT).} \end{cases}$$

4 Modified Lax hierarchies

Modified Lax hierarchies are obtained by simple gauge transformations $L \to L' = \Phi^{-1} L \circ \Phi$, where Φ is an eigenfunction of L. [4] The new operator satisfies

$$L'_{t_n} = [(L'^n)_{\geq 1}, L'] \,, \tag{10}$$

where $L' = u'_k \Delta^k + u'_{k-1}\Delta^{k-1} + \cdots$ has to be sorted with respect to powers of Δ and $(.)_{\geq 1}$ denotes the projection to positive powers of Δ. Dressing operators now need to satisfy the modified hierarchy

$$W'_{t_n} = -\, W'\,(L'^n)_{\geq 1} \;+\; (W'L'^n W'^{-1})_{\geq 1} W' \tag{11}$$

in order to preserve (10). Solutions of (11) can be derived systematically from the previous results. We note that the modified eigenfunction dynamics $\phi'_{t_n} = (L'^n)_{\geq 1}\phi'$ admits the trivial function 1. The DTs W of Theorem 1, followed by a gauge transformation with the image $\Phi' = W1$ (given by Theorem 2), can be used to generate dressing operators $W' = \Phi'^{-1} W$ satisfying (11). One finds the modified DTs $L' \to W'L'W'^{-1}$ triggered by (adjoint) eigenfunctions ϕ'_0 and ψ'_0 of L':

$$W' = \begin{cases} (\Delta\,\phi_0'^{-1})^{-1}\,\Delta \circ \phi_0'^{-1} & \text{(modified DT)} \\ \omega'^{-1}\,\Delta^{-1} \circ \psi_0'^{*} & \text{(modified adjoint DT)} \\ (1 - \phi'_0\,\Omega'^{-1}\,\omega')^{-1}\,(\,1 - \phi'_0\,\Omega'^{-1}\Delta^{-1} \circ \psi_0'^{*}\,) & \text{(modified binary DT)} \end{cases}$$

with $\Omega' = \Omega(\psi_0'^{*}, \phi'_0)$ and $\omega' = \Omega(\psi_0'^{*}, 1)$.

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