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Poisson Brackets for Integrable Lattice Systems

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In remembrance of our dear colleague Irene Dorfman.

Abstract

Poisson brackets associated with Lax operators of lattice systems are considered. Linear brackets originate from various r-matrices on the algebra of (pseudo-) shift symbols. Quadratic brackets are investigated which provide Hamiltonian formulations for various reductions of the (modified) Lattice KP hierarchy.

1 Introduction

The investigation of (multi-)Hamiltonian formulations probably is the simplest and most direct approach to integrable systems. The Hamiltonian version of Noether's theorem states that the conserved 1-forms are in unique correspondence with the generators of symmetries. The map between the invariant covectorfields (in particular, the gradients of conservation laws) and invariant vectorfields (symmetries) is given by the Poisson tensor associated with the Poisson bracket. If two distinct tensors $\mathcal{P}_{1,2}$ are known for an equation, then the simple recursive scheme $\mathcal{P}_1 \nabla H_i = \mathcal{P}_2 \nabla H_{i-1}$ may be used to construct a sequence of Hamiltonian functions H_i in involution. These constitute conservation laws for that equation, the associated Hamiltonian vectorfields are the symmetry generators. The bi-Hamiltonian structure was invented and investigated in the fundamental papers [1] by Irene Dorfman and Gelfand and independently by Magri [2] and Fuchssteiner and Fokas [3]. Subsequently much work by many authors has been devoted to this machinery: we refer to Irene's important monograph [4] for an overview of the theory and for references.

Nowadays one thinks of a Poisson bracket as a geometrical quantity associated with the underlying phase space rather than as a structure associated with individual equations. In the context of an integrable system it is natural to connect the bracket with its Lax representation $L_t = [A, L]$, where L lies in a suitable operator algebra. Indeed, for the celebrated KdV hierarchy and its generalizations the fundamental papers [5] by Gelfand and Dikii provided a construction based on the Lax operators which are to be regarded as elements of the algebra of pseudo-differential symbols. This approach also works in a context where bi-Hamiltonian formulations and recursion operators do not exist in the usual sense [6].

Generalizations of these results culminated in a Lie algebraic setting now referred to as the Adler-Kostant-Symes scheme [7], where the construction of an integrable hierarchy and its (first) Hamiltonian formulation originates solely from the properties of the underlying algebra. An interpretation in terms of classical r-matrices and a simple representation of a second ("quadratic") Poisson bracket was given by Semenov [8]. However, certain technical restrictions apply in the construction of the quadratic bracket. Semenov's original formula used "unitary" (skew-symmetric) r-matrices. This restriction was partially resolved in Refs. [9] and [10], where formulas for quadratic Poisson brackets were given for more general r-matrices:

$$\begin{array}{lcl} \mathcal{P}_1(L): \ \nabla H & \to & \frac{1}{2}\left([r(\nabla H), L] + r^*([\nabla H, L]) \right), \\ \mathcal{P}_2(L): \ \nabla H & \to & \frac{1}{4}\left([r(L\nabla H + \nabla H L), L] + Lr^*([\nabla H, L]) + r^*([\nabla H, L])L \right) \end{array} \tag{1.1}$$

are two compatible Poisson tensors that can be associated with Lax operators. Here L may be an element of an arbitrary associative algebra g endowed with a bi-invariant metric. The linear tensor $\mathcal{P}_1$ is a Poisson tensor for arbitrary r-matrices. Severe technical assumptions still apply to grant Poisson properties of the quadratic tensor: both r and its skew-symmetric part need to solve modified Yang-Baxter equations (for details see Refs. [9, 10], where also a cubic bracket was given).

These restrictions implied serious technical difficulties in the systematic construction of the quadratic brackets associated with such simple models as the modified KdV equation or the relativistic Toda equation [10].

A more general construction of quadratic brackets was recently proposed by Suris:

Theorem 1 [11]: *On any associative algebra $g = g^*$ equipped with a nondegenerate symmetric invariant metric $\langle a, bc\rangle = \langle ab, c\rangle = \langle c, ab\rangle$ the quadratic tensor*

$$\mathcal{P}_S(L) \ : \ \nabla H \ \to \ \frac{1}{2}\Big(A_1(L\nabla H)\, L - LA_2(\nabla H L) + S(\nabla H L)L - LS^*(L\nabla H) \Big) \tag{1.2}$$

defines a Poisson structure, if the linear maps $A_{1,2} : g \to g$ are skew-symmetric solutions of the modified Yang-Baxter equation

$$[A(\xi), A(\eta)] + [\xi, \eta] = A(\ [A(\xi), \eta] + [\xi, A(\eta)]\ ) \tag{1.3}$$

and the linear map $S : g \to g$ with adjoint S^* satisfies

$$\begin{array}{rcl} S(\ [A_2(\xi), \eta] + [\xi, A_2(\eta)]\ ) & = & [\ S(\xi), S(\eta)\ ], \\ S^*(\ [A_1(\xi), \eta] + [\xi, A_1(\eta)]\ ) & = & [S^*(\xi), S^*(\eta)] \end{array} \tag{1.4}$$

for all $\xi, \eta \in g$.

Since two r-matrices $A_{1,2}$ are involved, this result provides more flexibility in constructing suitable quadratic brackets on the underlying algebra of Lax operators. For many important examples the connection to the r-matrix giving rise to the first (linear) Poisson bracket is given by

$$r = A_1 + S = A_2 + S^* ,$$

i.e., the maps $A_1, \ldots, S^*$ originate from decompositions of the r-matrix. In this case the Poisson bracket

$$\begin{aligned} \{H_1, H_2\}(L) &= \langle \nabla H_2, \mathcal{P}_S(L)\nabla H_1\rangle \\ &= \langle L\nabla H_2, (A_1 + S)(L\nabla H_1)\rangle \; + \; \langle L\nabla H_2, S\,([\nabla H_1, L]\rangle \\ &\quad - \langle \nabla H_2 L, (A_2 + S^*)(\nabla H_1 L)\rangle \; + \; \langle \nabla H_2 L, S^*([\nabla H_1, L]\rangle \end{aligned}$$

vanishes for all pairs of Casimir functions H_i, (i.e., $[\nabla H_i, L] = 0$). The Casimirs form the set of commuting integrals for the integrable system under consideration. The Hamiltonian system associated with a Casimir C is the Lax equation

$$L_t = \mathcal{P}_S \nabla C = \tfrac{1}{2}\,[r(L\nabla C), L]\,.$$

Further, the following technical result is useful[1]:

Theorem 2: *Let $r = A_1 + S = A_2 + S^*$ satisfy the modified Yang-Baxter equation (1.3). If $A_{1,2}$ are skew symmetric w.r.t. the underlying bi-invariant metric, then (1.4) imply that both A_1 and A_2 satisfy (1.3).*

Proof: From the modified Yang-Baxter equation for $r = A_1 + S$ and using (1.4) one derives

$$\Delta(\xi, \eta) := [A_1(\xi), A_1(\eta)] + [\xi, \eta] - A_1([A_1(\xi), \eta] + [\xi, A_1(\eta)]) =$$

$$S([S^*(\xi), \eta] + [\xi, S^*(\eta)]) + A_1([S(\xi), \eta] + [\xi, S(\eta)]) - [S(\xi), A_1(\eta)] - [A_1(\xi), S(\eta)]\,.$$

The invariance of the metric and skew symmetry of A_1 leads to the identity

$$\begin{aligned} \langle\, \Delta(\xi, \eta), \theta\rangle &= \langle\, [S^*(\eta), S^*(\theta)] - S^*([A_1(\eta), \theta] + [\eta, A_1(\theta)]), \xi\rangle \\ &\quad + \langle\, [S^*(\theta), S^*(\xi)] - S^*([A_1(\theta), \xi] + [\theta, A_1(\xi)]), \eta\rangle \end{aligned}$$

with arbitrary $\xi, \eta, \theta \in g$. The corresponding identity holds for A_2 and S.

Q.E.D.

This observation reduces the verification of the technical condition (1.3) for $A_{1,2}$ to (1.4). The latter relations are often more easily checked.

In Section 2 we review the construction of Poisson brackets for reductions of the KP hierarchy and the modified KP hierarchy. Suris' bracket is crucial in these considerations.

[1] A related statement is given in Theorem 3 of Ref. [11].

In Section 3 discrete versions of these structures are presented generalizing Suris' original results which he applied in a discrete context. The underlying set of dynamical equations is the (modified) Lattice KP hierarchy, a systematic construction of the Poisson brackets associated with various constraints is given. Several Poisson brackets are naturally associated with some simple r-matrices. Results concerning the reduction properties of these brackets to finite Lax operators are presented. In particular, the Hamiltonian structures of the Toda model and the relativistic Toda model are described as restrictions of the same Poisson structure to different integral manifolds.

2 Review of Poisson brackets for continuous soliton equations

2.1 Poisson brackets for constrained KP flows

In their fundamental papers [5] Gelfand and Dikii considered the Hamiltonian structures of the soliton equations associated with (scalar) Lax operators of the type

$$L = \partial^N + u_{N-2}\partial^{N-2} + u_{N-1}\partial^{N-1} + \cdots + u_0 \ . \tag{2.1}$$

The corresponding hierarchies of generalized KdV equations for the fields $u_0, \ldots, u_{N-2}$ are encoded in the Lax equations

$$L_{t_q} = [(L^q)_{\geq 0}, L] \, , \tag{2.2}$$

where the subscript ≥ 0 denotes the projection of the pseudo-differential fractional powers L^q onto their nonnegative differential orders. In particular, the case $N = 2$ represents the KdV hierarchy proper, $N = 3$ yields the Boussinesq hierarchy etc. The bi-Hamiltonian formulations for these equations arise from two abstract Poisson brackets associated with the Lax operator which are given by the following compatible Poisson tensors

$$\begin{array}{lcll} \mathcal{P}_1(L): & \nabla H & \to & [L, (\nabla H)_{<0}]_{\geq 0} \, , \\ \mathcal{P}_2(L): & \nabla H & \to & (L\,\nabla H)_{\geq 0} L - L\,(\nabla H\, L)_{\geq 0} \, . \end{array} \tag{2.3}$$

The linear tensor is the Lie Poisson structure of the Adler-Kostant-Symes scheme [7], whereas $\mathcal{P}_2$ corresponds to Semenov's quadratic bracket. Indeed, consider the algebra of pseudo-differential symbols

$$g = g_{\geq 0} \oplus g_{<0} = \Big\{ \sum_{n \geq 0} u_n \partial^n \Big\} \oplus \Big\{ \sum_{n<0} u_n \partial^n \Big\}$$

with the r-matrix $r(\xi) = \xi_{\geq 0} - \xi_{<0}$. Due to the unitarity (skew-symmetry) of r with respect to Adler's trace duality [7]

$$\langle \xi^*, \xi \rangle = \mathrm{tr}(\xi^* \xi) \, , \quad \mathrm{tr}(\xi) = \mathrm{tr}\Big(\textstyle\sum_n u_n \partial^n \Big) := \int u_{-1}(x) \, dx$$

both tensors (1.1) represent compatible Poisson brackets on g. These maps are nothing but (2.3). For the Casimir functions $H = \mathrm{tr}(L^{q+1})/(q+1)$ (i.e., $\nabla H = L^q$) the associated Hamiltonian vector fields are the Lax equations (2.2), which give rise to the KP hierarchy for a general pseudo-differential $L \in g$. The restriction to Lax operators of the form (2.1) represents a reduction to the integrable equations of the Gelfand-Dikii hierarchy in 1+1 dimensions for which the bi-Hamiltonian formulations are given by (1.1)/(2.3). The linear Poisson tensor admits a proper restriction to operators of the form (2.1), where $(1.1.\mathcal{P}_1)$ yields $(2.3.\mathcal{P}_1)$. The quadratic tensor will in general produce a term of order ∂^{N-1}, so that a term $u_{N-1}\partial^{N-1}$ has to be included in (2.1). Dirac reduction has to be used to generate a bracket preserving the constraint $u_{N-1} = 0$. Usually such a reduction procedure gives rise to nonlocal terms (integrations) in the Poisson structure. In the present case, however, this does not happen. This is explained by the following representation

$$\mathcal{P}_2^{(red)}(L) : \nabla H \to (L\nabla H)_{\geq 0}L - L(\nabla H L)_{\geq 0} + \tfrac{1}{N}\left[D^{-1}(\mathrm{res}([\nabla H, L])), L\right] \quad (2.4)$$

of the reduced tensor, where $\mathrm{res}(\xi) = \mathrm{res}(\sum_n u_n\partial^n) = u_{-1}$ is the usual residue and $D^{-1} = \frac{1}{2}(\int^x - \int_x)$ is an integration stemming from the reduction. As the residue of a commutator is in the image of the differential operator [7], the formal integration yields some purely differential expression of the coordinates, wence the reduced bracket is again local. It provides the "second" Hamiltonian operator of the Gelfand-Dikii hierarchies. We note the explicit dependence of $\mathcal{P}_2^{(red)}$ on N, through which these tensors can be restricted properly to Nth order operators with $u_N = 1$ and $u_{N-1} = 0$ (a Poisson submanifold).

Suris' result as presented in Theorem 1 leads to a direct construction of (2.4). With

$$A_1 = A_2 = r : \xi \to \xi_{\geq 0} - \xi_{<0}, \quad S = S^* = 0$$

the tensor $\mathcal{P}_S(L)$ coincides with $(1.1.\mathcal{P}_2)/(2.3.\mathcal{P}_2)$. Alternatively, with

$$A_1 : \xi \to \xi_{\geq 0} - \xi_{<0} - \tfrac{1}{N} D^{-1}(\mathrm{res}(\xi)), \quad S = \tfrac{1}{N} D^{-1}(\mathrm{res}(\xi)),$$

$$A_2 : \xi \to \xi_{\geq 0} - \xi_{<0} + \tfrac{1}{N} D^{-1}(\mathrm{res}(\xi)), \quad S^* = -\tfrac{1}{N} D^{-1}(\mathrm{res}(\xi))$$

the tensor $\mathcal{P}_S$ produces the reduced Poisson structure (2.4). In all cases one has

$$r = A_1 + S = A_2 + S^* : \quad \xi \to \xi_{\geq 0} - \xi_{<0}, \quad (2.5)$$

where the r-matrix r trivially satisfies the modified Yang-Baxter equation, since it stems from a Lie algebra decomposition. The technical assumptions (1.4) are easily checked.

Recently there has been much interest in reductions of the KP hierarchy via the symmetry generated by squared eigenfunctions (e.g., [12]–[17]). It was found that the resulting constrained KP flows (2.2) are characterized by pseudo-differential Lax operators of the form

$$L = \partial^N + u_{N-2}\partial^{N-2} + \cdots + u_0 + \sum_{i=1}^{K} \phi_i \partial^{-1}\psi_i, \quad (2.6)$$

which constitute an invariant operator manifold for the KP dynamics. The evolution (2.2) implies that ϕ_i and ψ_i are eigenfunctions and adjoint eigenfunctions of the KP hierarchy satisfying

$$\phi_{it_q} = ((L^q)_{\geq 0}\phi_i)\,, \quad \psi_{it_q} = -(((L^q)_{\geq 0})^*\psi_i)\,, \quad i = 1, \ldots, K\,. \tag{2.7}$$

These, however, are not linear equations, since the fields ϕ_i, ψ_i enter nonlinearly via L^q. The corresponding $1+1$-dimensional hierarchies for the fields $u_0, \ldots, u_{N-2}, \phi_1, \ldots, \phi_K, \psi_1, \ldots, \psi_K$ are bi-Hamiltonian. Indeed, the operators (2.6) form a proper Poisson submanifold for both the linear tensor ($1.1.\mathcal{P}_1$) and the quadratic tensor (2.4) associated with the standard r-matrix (2.5). A simple restriction to (2.6) provides the Hamiltonian structures. Explicit formulas can be found in Ref. [18].

2.2 Poissonbracket for constrained modified KP flows

In Ref. [19] the bi-Hamiltonian construction was extended to reductions of the "nonstandard" Lax equations [20] given by

$$L_{t_q} = [(L^q)_{\geq 1}, L]\,, \tag{2.8}$$

where – in contrast to the KP case – zero order terms are excluded from $(L^q)_{\geq 1}$. In the remainder of this section we review the results of Ref. [19].

For a generic pseudo-differential $L = \partial + v_0 + v_{-1}\partial^{-1} + \cdots$ the resulting equations for the field v_0 constitute the $2+1$-dimensional hierarchy of modified KP flows [21, 22, 23], so that (2.8) shall be referred to as the modified KP hierarchy. These equations have been studied extensively in Refs. [20]–[24], where also a further class of integrable Dym-type equations is considered.

It is easily verified that the modified dynamics (2.8) may be restricted to operator manifolds of the form

$$L = \partial^N + v_{N-1}\partial^{N-1} + v_{N-2}\partial^{N-2} + \cdots + v_1\partial \tag{2.9}$$

or

$$L = \partial^N + v_{N-1}\partial^{N-1} + v_{N-2}\partial^{N-2} + \cdots + v_1\partial + v_0 + \partial^{-1}\Psi \tag{2.10}$$

or

$$L = \partial^N + v_{N-1}\partial^{N-1} + v_{N-2}\partial^{N-2} + \cdots + v_1\partial + v_0 + \partial^{-1}\Psi + \sum_{i=1}^{K} \phi_i\partial^{-1}\psi_i\,. \tag{2.11}$$

Operators of the form (2.9) arise from gauge transformations $\tilde{L} = \Phi^{-1}L \circ \Phi$, where L is given by (2.1). Indeed, if Φ is an eigenfunction satisfying $L\Phi = 0$, then $\tilde{L}$ is of the form (2.9). Further, if Φ satisfies $\Phi_{t_q} = ((L^q)_{\geq 0}\Phi)$, then the dynamics (2.2) yields (2.8) for $\tilde{L}$ (see e.g. Ref. [23]). Similarly, operators of the form (2.10)/(2.11) arise from gauge transformations of the operators (2.6)[18].

The underlying r-matrix is again built on the algebra g of pseudo-differential symbols, this time with a decomposition $g = g_{\geq 1} \oplus g_{<1}$ into strictly positive differential orders versus nonpositive orders. The resulting linear map $\tilde{r} : \xi \to \xi_{\geq 1} - \xi_{<1}$ is again an r-matrix, since both $g_{\geq 1}$ as well as $g_{<1}$ are sub-(Lie)algebras of g. The relevance of this "nonstandard" decomposition in the context of soliton systems had already been observed by Reiman [25].

The linear tensor (1.1.$\mathcal{P}_1$) with $\tilde{r}$ gives rise to the Poisson tensor

$$\begin{aligned} \tilde{\mathcal{P}}_1(L) \, : \, \nabla H \;\to\; & [\nabla H, L]_{<-1} - [(\nabla H)_{<1}, L] \\ & = [(\nabla H)_{\geq 1}, L]_{<-1} - [(\nabla H)_{<1}, L]_{\geq -1} \end{aligned} \tag{2.12}$$

which reduces to $\tilde{\mathcal{P}}_1(L) = [L, (\nabla H)_{<1}]_{\geq -1}$ for $L \in g_{\geq -1} = (g_{<1})^*$, i.e., for operators of the form (2.10). The tensor (2.12) provides a first Hamiltonian formulation for the flows (2.8) with the Hamiltonians $H = \mathrm{tr}(L^{q+1})/(q+1)$. The fact that the modified KP dynamics can be restricted to operators of the form (2.10) and (2.11), respectively, is explained by its Hamiltonian nature: these operators constitute integral manifolds of the linear bracket.

Theorem 3 [19]: *The linear Poisson tensor (2.12) admits a proper restriction to Lax operators of the form (2.10) and (2.11).*

However, no proper Hamiltonian restriction to operators of the form (2.9) is possible. Dirac reduction has to be invoked to obtain a Poisson bracket for the integrable hierarchies associated with the Lax operators (2.9).

With these considerations a first Hamiltonian structure for the equations (2.8) has been established via the linear tensor (2.12).

In the construction of a quadratic Poisson bracket a serious problem arises. As discussed in Ref. [10], Semenov's quadratic tensor (1.1.$\mathcal{P}_2$) does *not* provide a Poisson bracket for the modified r-matrix $\tilde{r}$, since it is neither skew-symmetric nor does its skew-symmetric part satisfy the Yang-Baxter equation. However, the more general construction of Theorem 1 is capable of producing suitable brackets. It can be verified that the maps

$$\begin{aligned} A_1 \, : \, \xi \;&\to\; \xi_{\geq 1} - \xi_0 + \partial^{-1}\mathrm{res}(\xi) - \xi_{<-1} - 2\, D^{-1}(\mathrm{res}(\xi)) \,, \\ A_2 \, : \, \xi \;&\to\; \xi_{\geq 0} - \xi_{<0} + 2\, D^{-1}(\mathrm{res}(\xi)) \,, \\ S \, : \, \xi \;&\to\; -2\, \partial^{-1}\mathrm{res}(\xi) + 2\, D^{-1}(\mathrm{res}(\xi)) \,, \\ S^* \, : \, \xi \;&\to\; -2\, \xi_0 - 2\, D^{-1}(\mathrm{res}(\xi)) \end{aligned} \tag{2.13}$$

formally satisfy the assumptions of Theorem 1. Here $\xi_0, \xi_{<-1}$ etc. are the terms of differential order 0, less than -1 etc. We note

$$\tilde{r} = A_1 + S = A_2 + S^* : \quad \xi \;\to\; \xi_{\geq 1} - \xi_{<1} \,,$$

so that according to Theorem 2 only the technical conditions (1.4) need to be checked. The tensor (1.2) results in

$$\begin{aligned} \tilde{\mathcal{P}}_S(L) \, : \, \nabla H \;\to\; & (L\nabla H)_{\geq 1} L - L(\nabla H\, L)_{\geq 0} + L(L\, \nabla H)_0 \\ & - \partial^{-1}\mathrm{res}([\nabla H, L])\, L + [D^{-1}(\mathrm{res}([\nabla H, L]), L] \end{aligned} \tag{2.14}$$

which was found in Ref. [18]. Swapping A_1 and A_2 and simultaneously replacing S by S^* the tensor (1.2) results in

$$\begin{aligned}\tilde{\mathcal{P}}'_S(L) \; : \; \nabla H \;\rightarrow\; & (L\,\nabla H)_{\geq 0}L - L\,(\nabla H\; L)_{\geq 1} - (\nabla H\; L)_0\; L \\ & -L\,\partial^{-1}\mathrm{res}([\nabla H, L]) - [D^{-1}(\mathrm{res}([\nabla H, L]), L]\,.\end{aligned} \tag{2.15}$$

Both $\tilde{\mathcal{P}}_S$ and $\tilde{\mathcal{P}}'_S$ are compatible with (2.12). This follows from the usual argument[2] that the deformation $L \rightarrow L + \epsilon 1$ of both quadratic tensors produces the linear tensor as the term linear in ϵ. Both $\tilde{\mathcal{P}}_S$ and $\tilde{\mathcal{P}}'_S$ play a role as quadratic Hamiltonian structures for the dynamics (2.8):

$$L_{t_q} = [(L^q)_{\geq 1}, L] = \tilde{\mathcal{P}}_S(L)\;\nabla H = \tilde{\mathcal{P}}'_S(L)\;\nabla H\,, \quad H(L) = \mathrm{tr}(L^q)/q\,.$$

However, when restricting these equations to Lax operators of a specified form, the two tensors behave differently:

Theorem 4 [19]: *The tensor (2.14) admits a proper restriction to Lax operators of the form (2.10) and (2.11).*

No proper restriction to (2.9) is possible. Dirac reduction has to be invoked to obtain the quadratic bracket associated with the constraint $v_0 = \Psi = 0$. It turns out that the reduction of (2.14) to the submanifold (2.9) is identical with (2.15):

Theorem 5 [19]: *The tensor (2.15) admits a proper restriction to Lax operators of the form (2.9), where it coincides with the Dirac reduction of (2.14).*

We note that the integration D^{-1} is not well-defined in (2.13), so that the argument remains formal. However, in (2.14) as well as in (2.15) this problem does not arise, as the residues of the commutators can be integrated properly. The resulting Poisson brackets are local.

3 Lattice systems

It was shown in Ref. [11] that a unified description of the Hamiltonian structures of the Toda model as well as the relativistic Toda model is provided by the construction of Theorem 1. We return to Suris' original considerations in a slightly more general context and generalize the results of Ref. [11]. Each of the results presented in the previous section has a discrete analog.

3.1 Notation

We consider the algebra of shift operators

$$g = \left\{ u_N(n)T^N + u_{N-1}(n)T^{N-1} + \cdots + u_0(n) + u_{-1}(n)T^{-1} + \cdots \right\}$$

[2] See also Theorem 3 of Ref. [11]: with $\tilde{r} = A_1 + S = A_2 + S^*$ Suris' quadratic bracket is automatically compatible with the linear bracket generated by $\tilde{r}$.

with commuting coefficients u_j from some commutative algebra g_0. They depend on a discrete lattice index n. The action of the shift symbols on functions (of n) is given by $(T^k u)(n) = u^{(k)}(n) := u(n+k)$. Projections to various shift orders are denoted by

$$\xi_j = u_j T^j \,, \quad \xi_{\geq k} = \sum_{j \geq k} u_j T^j \,, \quad \xi_{<k} = \sum_{j<k} u_j T^j$$

for $\xi = \sum_j u_j T^j \in g$. The (pseudo-) difference symbol

$$\Delta^{-1} = T^{-1} + T^{-2} + \cdots$$

is introduced which may be regarded as the formal inverse of the difference operator $\Delta = T - 1$.

We use the trace form $\mathrm{tr}(\xi) = \sum_n \xi_0(n)$ to identify g and its dual g^* via $\langle \xi^*, \xi \rangle = \mathrm{tr}(\xi^* \xi)$. This metric is bi-invariant due to the symmetry of the trace. Indeed, the zero order term of a commutator is in the image of the difference operator:

$$\begin{aligned} [\xi, \eta]_0 &= \sum_j \Big(u_j v_{-j}^{(j)} - v_j u_{-j}^{(j)} \Big) = \sum_{j>0} \Big(u_j v_{-j}^{(j)} + u_{-j} v_j^{(-j)} - v_j u_{-j}^{(j)} - v_{-j} u_j^{(-j)} \Big) \\ &= \sum_{j>0} \Big((T^j - 1)(v_{-j} u_j^{(-j)} - u_{-j} v_j^{(-j)}) \Big) = \Big((T-1) s \Big) \end{aligned}$$

with $\xi = \sum_j u_j T^j$, $\eta = \sum_j v_j T^j$ and

$$s = \sum_{j>0} \Big((1 + T + \cdots + T^{j-1})(v_{-j} u_j^{(-j)} - u_{-j} v_j^{(-j)}) \Big) \,.$$

Consequently, $\mathrm{tr}(\xi\eta - \eta\xi) = \sum_n [\xi, \eta]_0 = \sum_n \Big((T-1)s \Big) = 0 \,.$

3.2 The Lattice KP hierarchy and its reductions

We consider the hierarchy of commuting Lax equations [26]

$$\begin{aligned} L_{t_q} &= \ [\, (L^q)_{\geq 1} + \tfrac{\alpha+1}{2} \, (L^q)_0 \,,\, L \,] \\ &= -[\, (L^q)_{<0} + \tfrac{1-\alpha}{2} \, (L^q)_0 \,,\, L \,] \end{aligned} \tag{3.1}$$

with $L \in g$ and some scalar parameter α. The corresponding set of nonlinear differential-difference equations shall be called the Lattice KP hierarchy. The underlying r-matrix is

$$r_\alpha(\xi) = \xi_{\geq 1} + \alpha \, \xi_0 - \xi_{<0}$$

which originates from a "triangular" decomposition $g = g_{\geq 1} \oplus g_0 \oplus g_{<0}$ of the operators into "strictly upper", diagonal and "strictly lower" parts. We note that simple r-matrices arise from such a decomposition, defined as (minus) the identity on the strictly upper (lower) subalgebra. The action on the diagonal may be given by an arbitrary r-matrix on g_0 which, in the present case, is just multiplication by α.

One obtains a family of linear Poisson brackets (1.1.$\mathcal{P}_1$):

$$\begin{aligned} &\mathcal{P}_1(L): \ \nabla H \ \rightarrow \\ &\qquad [\nabla H_{\geq 1}, L] - [\nabla H, L]_{\geq 1} + \tfrac{\alpha+1}{2}[\nabla H_0, L] + \tfrac{\alpha-1}{2}[\nabla H, L]_0 \\ &= -[\nabla H_{<0}, L] + [\nabla H, L]_{<0} + \tfrac{\alpha-1}{2}[\nabla H_0, L] + \tfrac{\alpha+1}{2}[\nabla H, L]_0 \,. \end{aligned} \tag{3.2}$$

The Casimir function $H = \mathrm{tr}(L^{q+1})/(q+1)$ with $\nabla H = L^q$ produces (3.1) as associated Hamiltonian system, so that $\mathcal{P}_1$ provides the first Hamiltonian formulation for these equations. The dynamics admits a Hamiltonian restriction to finite Lax operators:

Theorem 6: *The linear tensor (3.2) admits a proper restriction to Lax operators of the form*

$$L = u_N T^N + \cdots + u_M T^M + \sum_{i=1}^{K} \phi_i T^M \Delta^{-1} \psi_i \,, \quad N \geq 0 \geq M \,. \tag{3.3}$$

Proof: From the second representation of (3.2) it is clear that the highest shift order of $\mathcal{P}_1(L)\nabla H$ coincides with the highest shift order of L, provided it is nonnegative. We note the identities

$$(A\phi\, T^M \Delta^{-1})_{<M} = (A\phi)\, T^M \Delta^{-1} \,, \quad (T^M \Delta^{-1} \psi A)_{<M} = T^M \Delta^{-1} (A^* \psi) \tag{3.4}$$

which hold for arbitrary functions ϕ, ψ and symbols $A \in g_{\geq 0}$. From the first representation of (3.2) one computes for $M \leq 0$

$$\begin{aligned} (\mathcal{P}_1(L)\nabla H)_{<M} &= [\, B \,, \textstyle\sum_i \phi_i\, T^M \Delta^{-1} \psi_i \,]_{<M} \\ &\overset{(3.4)}{=} \textstyle\sum_i (B\phi_i)\, T^M \Delta^{-1} \psi_i - \sum_i \phi_i\, T^M \Delta^{-1} (B^* \psi_i) \end{aligned}$$

with $B = \nabla H_{\geq 1} + \frac{\alpha+1}{2} \nabla H_0 \in g_{\geq 0}$, so that $\mathcal{P}_1(L)\nabla H$ is tangent to L.

Q.E.D.

From the proof it is clear that the functions ϕ_i, ψ_i inherit the dynamics

$$\phi_{i t_q} = (B\phi_i) \,, \quad \psi_{i t_q} = -(B^* \psi_i) \,, \quad B = \nabla H_{\geq 1} + \tfrac{\alpha+1}{2} \nabla H_0 \,.$$

The parameter values $\alpha = \pm 1$ are special in that the corresponding r-matrices stem from simple Lie algebra decompositions $g = g_{\geq 0} \oplus g_{<0}$ ($\alpha = 1$) and $g = g_{\geq 1} \oplus g_{<1}$ ($\alpha = -1$), respectively. Both values lead to the same nonlinear equations since transposition of the Lax operators provides the link between the cases $\alpha = 1$ and $\alpha = -1$:

$$\begin{aligned} &L_{t_q} = [(L^q)_{\geq 0}, L] \quad \Longrightarrow \\ &\qquad L^*_{t_q} = -[((L^q)_{\geq 0})^*, L^*] = -[(L^{*q})_{\leq 0}, L^*] = [(L^{*q})_{\geq 1}, L^*] \,. \end{aligned}$$

For $\alpha = 1$ the highest coefficient in the Lax operator may be constrained. By transposition, the same is true for the lowest coefficient (if existent) for $\alpha = -1$:

Theorem 7:

a) For $\alpha = 1$ and $N \geq 1 \geq M$ the linear tensor (3.2) admits a proper restriction to Lax operators of the form

$$L = T^N + u_{N-1}T^{N-1} + \cdots + u_M T^M + \sum_{i=1}^{K} \phi_i T^M \Delta^{-1} \psi_i \,. \qquad (3.5)$$

b) For $\alpha = -1$, $N > M$ and $0 > M$ the linear tensor (3.2) admits a proper restriction to Lax operators of the form

$$L = u_N T^N + \cdots + u_{M+1} T^{M+1} + T^M \,.$$

These statements are easily verified with the first and second representation of (3.2).

We now turn to a discussion of the quadratic Poisson brackets that may be associated with the Lax equations (3.1). We note that the skew-symmetric part

$$\tfrac{1}{2}\,(r_\alpha - r_\alpha^*) = r_0 : \ \xi \ \to \ \xi_{\geq 1} - \xi_{<0}$$

of the r-matrix solves the modified Yang-Baxter equation (1.3), so that Semenov's quadratic tensor (1.1.$\mathcal{P}_2$)

$$\begin{aligned} \mathcal{P}_2(L) : \ \nabla H \ \to \ & (L\nabla H)_{\geq 1} L - L(\nabla H L)_{\geq 1} + \tfrac{1}{2}\,(L\nabla H)_0 L - \tfrac{1}{2}\,L(\nabla H L)_0 \\ & \qquad + \tfrac{\alpha}{2}\,(\nabla H L)_0 L - \tfrac{\alpha}{2}\,L(L\nabla H)_0 \\ = & -(L\nabla H)_{<0} L + L(\nabla H L)_{<0} - \tfrac{1}{2}\,(L\nabla H)_0 L + \tfrac{1}{2}\,L(\nabla H L)_0 \\ & \qquad + \tfrac{\alpha}{2}\,(\nabla H L)_0 L - \tfrac{\alpha}{2}\,L(L\nabla H)_0 \end{aligned} \qquad (3.6)$$

is indeed a Poisson tensor on g. The Casimir function $H = \mathrm{tr}(L^q)/q$ yields (3.1) as the associated Hamiltonian equation. The compatibility with the linear bracket is provided by the usual deformation $L \to L + \epsilon 1$. For any value of α a proper restriction to finite Lax operators of the form (3.3) is possible:

Theorem 8: *Semenov's quadratic Poisson tensor (3.6) admits a proper restriction to Lax operators of the form (3.3).*

The proof is straightforward using the two representations (3.6) and an argument similar to that in the proof of Theorem 6.

For the case $\alpha = 1$ the constraint $u_N = 1$ is not left invariant by the tensor (3.6), whence Dirac reduction has to be invoked to obtain a quadratic Poisson bracket for Lax operators of the form (3.5). Despite the reduction process the result is a local Hamiltonian structure. For a direct approach to the reduced bracket we use Theorem 1 to generalize (3.6) for arbitrary α.

Theorem 9: *Let $\rho : g_0 \to g_0$ be an arbitrary skew-symmetric linear map on the algebra g_0 of zero order terms of g. Then*

$$A_1 : \xi \to \xi_{\geq 1} - \xi_{<0} - \rho(\xi_0)\,, \quad S : \xi \to \alpha\,\xi_0 + \rho(\xi_0)\,,$$

$$A_2 : \xi \to \xi_{\geq 1} - \xi_{<0} + \rho(\xi_0)\,, \quad S^* : \xi \to \alpha\,\xi_0 - \rho(\xi_0)$$

satisfy all conditions of Theorem 1. The resulting Poisson tensor is

$$\mathcal{P}_S(L)\nabla H = \mathcal{P}_2(L)\nabla H + \tfrac{1}{2}\,[\,\rho([\nabla H, L]_0)\,, L\,]\,,$$

where $\mathcal{P}_2(L)$ is Semenov's quadratic tensor (3.6). It is compatible with the linear tensor (3.2) and admits a proper restriction to Lax operators of the form (3.3).

The technical conditions (1.4) are easily verified. With

$$A_1 + S = A_2 + S^* = r_\alpha : \xi \to \xi_{\geq 1} + \alpha\,\xi_0 - \xi_{<0}$$

this grants the Poisson properties of $\mathcal{P}_S$ according to Theorem 2. We note that the additonal term generated by ρ drops out for Casimir functions, so that $\mathcal{P}_S$ generates the same integrable hierarchies as $\mathcal{P}_2$.

In the following we shall concentrate on the case $\alpha = 1$. The two Poisson tensors under consideration are

$$\begin{aligned} \mathcal{P}_1(L) : \nabla H \;\to\; & [\nabla H_{\geq 0}, L] - [\nabla H, L]_{\geq 1} \\ =\; & -[\nabla H_{<0}, L] + [\nabla H, L]_{<1} \end{aligned} \qquad (3.7)$$

and

$$\mathcal{P}_S(L) : \nabla H \to \qquad (3.8)$$

$$(L\nabla H)_{\geq 1}L - L(\nabla H L)_{\geq 1} + \tfrac{1}{2}\,(\,[(L\nabla H + \nabla H L)_0 + \rho([\nabla H, L]_0), L]$$

$$= -(L\nabla H)_{<0}L + L(\nabla H L)_{<0} + \tfrac{1}{2}\,(\,L[\nabla H, L]_0 + [\nabla H, L]_0 L + [\rho([\nabla H, L]_0), L]\,)$$

which are compatible for any skew ρ. Since the associated integrable hierarchies $L_{t_q} = [(L^q)_{\geq 0}, L]$ (i.e., (3.1) with $\alpha = 1$) can be restricted to (3.5), one would like both $\mathcal{P}_1$ and $\mathcal{P}_S$ to admit a proper restriction to such Lax operators. For $\mathcal{P}_1$ this is granted by Theorem 7.a). For $\mathcal{P}_S$ the invariance of the constraint $u_N = 1$ can be achieved by an appropriate choice of ρ:

Theorem 10: *With $\rho(\xi_0) = ((T^N + 1)(T^N - 1)^{-1}\xi_0)$ the Poisson tensor (3.8) can be restricted properly to Lax operators of the form (3.5), where it coincides with the Dirac reduction of Semenov's tensor (3.6, $\alpha = 1$).*

Proof: According to Theorem 9 operators of the form (3.3) constitute an integral manifold of $\mathcal{P}_S$, whence it suffices to show that the constraint $u_N = 1$

is left invariant. From the second representation (3.8) the Nth shift order of $\mathcal{P}_S(L)\nabla H$ vanishes ($\xi_0 = [\nabla H, L]_0$):

$$T^N \xi_0 + \xi_0\, T^N + [\rho(\xi_0), T^N] = (\, \xi_0^{(N)} + \xi_0 + \rho(\xi_0) - \rho^{(N)}(\xi_0)\,)\, T^N = 0\,.$$

For the Dirac reduction of (3.6) induced by the constraint $u_N = 1$ one computes

$$((\mathcal{P}_2(L)\nabla H)_N = S(h^{(-N)} - h + [\nabla H_{>-N}, L]_0)\, T^N$$

with $S : \xi_0 \to \frac{1}{2}\,(\xi_0 + \xi_0^{(N)})$ and $\nabla H = \nabla H_{>-N} + T^{-N} h$ (i.e., $h = \frac{\delta H}{\delta u_N}$). Further,

$$\begin{aligned}(\mathcal{P}_2(L)\nabla H)_{<N} &= (\mathcal{P}_2(L)\nabla H_{>-N})_{<N} + (\mathcal{P}_2(L) T^{-N} h)_{<N} \\ &= (\mathcal{P}_2(L)\nabla H_{>-N})_{<N} + \tfrac{1}{2}\,[h^{(-N)} + h, L]_{<N}\,.\end{aligned}$$

The reduced tensor is obtained from elimination of h, when the constraint $\dot u_N = (\mathcal{P}_2(L)\nabla H)_N = 0$ is imposed. Insertion of $h = ((1 - T^{-N})^{-1}[\nabla H_{>-N}, L]_0)$ into $(\mathcal{P}_2(L)\nabla H)_{<N}$ yields the reduced tensor

$$\begin{aligned}\mathcal{P}_2^{(red)}(L)\nabla H_{>-N} &= ((\mathcal{P}_2(L)\nabla H)_{<N} \\ &= (\mathcal{P}_2(L)\nabla H_{>-N})_{<N} + \tfrac{1}{2}\,[\left(\tfrac{1+T^{-N}}{1-T^{-N}}\,[\nabla H_{>-N}, L]_0\right),\, L\,]_{<N} \\ &= (\mathcal{P}_S(L)\nabla H_{>-N})_{<N}\,.\end{aligned}$$

Since $\mathcal{P}_S(L)$ does not produce any terms of order $\geq N$, the final projection to orders $< N$ may be omitted: $\mathcal{P}_2^{(red)} = \mathcal{P}_S$.

Q.E.D.

Example 11: We consider the Lax operator $L = T + b + T^{-1}a$ of the celebrated Toda lattice hierarchy. The first equation $L_{t_1} = [L_{\geq 0}, L]$ is the Toda system itself:

$$a_{t_1} = a\,(b^{(1)} - b)\,, \quad b_{t_1} = a - a^{(-1)}\,. \tag{3.9}$$

In terms of the coordinates a and b the dual fibers are parametrized by $\nabla H = \frac{\delta H}{\delta b} + \frac{\delta H}{\delta a}\, T$, so that

$$\mathrm{tr}(\nabla H\, \dot L) = \mathrm{tr}\left((\tfrac{\delta H}{\delta b} + \tfrac{\delta H}{\delta a}\, T)\ (\dot b + T^{-1}\dot a)\right) = \textstyle\sum_n \frac{\delta H}{\delta b}\, \dot b + \sum_n \frac{\delta H}{\delta a}\, \dot a\,.$$

Insertion of ∇H into (3.7) and (3.8) yields the two Hamiltonian structures associated with (3.9):

$$\begin{aligned}\mathcal{P}_1(L)\nabla H &= a\,\tfrac{\delta H}{\delta a} - \left(a\,\tfrac{\delta H}{\delta a}\right)^{(-1)} + T^{-1}a\left((\tfrac{\delta H}{\delta b})^{(1)} - \tfrac{\delta H}{\delta b}\right)\,, \\ \mathcal{P}_S(L)\nabla H &= a\,(\tfrac{\delta H}{\delta b})^{(1)} - \left(a\,\tfrac{\delta H}{\delta b}\right)^{(-1)} + ba\,\tfrac{\delta H}{\delta a} - b\left(a\,\tfrac{\delta H}{\delta a}\right)^{(-1)} \\ &\quad + T^{-1}a\left(\left(a\,\tfrac{\delta H}{\delta a}\right)^{(1)} - \left(a\,\tfrac{\delta H}{\delta a}\right)^{(-1)} + \left(b\,\tfrac{\delta H}{\delta b}\right)^{(1)} - b\,\tfrac{\delta H}{\delta b}\right)\,,\end{aligned}$$

i.e.,

$$\mathcal{P}_1(L): \begin{pmatrix} \frac{\delta H}{\delta a} \\ \frac{\delta H}{\delta b} \end{pmatrix} \to \begin{pmatrix} 0 & a\,(T-1) \\ (1-T^{-1})\,a & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta a} \\ \frac{\delta H}{\delta b} \end{pmatrix},$$

$$\mathcal{P}_S(L): \begin{pmatrix} \frac{\delta H}{\delta a} \\ \frac{\delta H}{\delta b} \end{pmatrix} \to \begin{pmatrix} a\,(T-T^{-1})\,a & a\,(T-1)\,b \\ b\,(1-T^{-1})\,a & a\,T-T^{-1}\,a \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta a} \\ \frac{\delta H}{\delta b} \end{pmatrix}.$$

The Hamiltonians of (3.9) are

$$H_1 = \mathrm{tr}(L) = \sum_n b \quad \text{and} \quad H_2 = \tfrac{1}{2}\,\mathrm{tr}(L^2) = \sum_n (a + \tfrac{1}{2}\,b^2)\,.$$

Example 12: We consider Lax operators of the form (3.5), $N = M = K = 1$:

$$L = T + \phi T \Delta^{-1} \psi\,. \tag{3.10}$$

The associated hierarchy of Lax equations $L_{t_q} = [(L^q)_{\geq 0}, L]$ is equivalent to

$$\phi_{t_q} = ((L^q)_{\geq 0}\phi)\,, \quad \psi_{t_q} = -(((L^q)_{\geq 0})^* \psi)\,.$$

The basic system

$$\phi_{t_1} = \phi^{(1)} + \psi\phi^2\,, \quad \psi_{t_1} = -\psi^{(-1)} - \phi\psi^2 \tag{3.11}$$

is related to the relativistic Toda model (see Example 18). The dual fibers of the manifold (3.10) with coordinates ϕ, ψ may be parametrized by operators $\nabla H \in g$ with

$$((\nabla H_{\geq 0})^* \psi) = \tfrac{\delta H}{\delta \phi}\,, \quad (\nabla H_{\geq 0}\phi) = \tfrac{\delta H}{\delta \psi}\,,$$

so that via (3.4)

$$\begin{aligned} (T\Delta^{-1}\psi\nabla H)_0 &= (T\Delta^{-1}((\nabla H_{\geq 0})^*\psi))_0 = \tfrac{\delta H}{\delta \phi}\,, \\ (\nabla H \phi T \Delta^{-1})_0 &= \quad ((\nabla H \phi) T\Delta^{-1})_0 \quad = \tfrac{\delta H}{\delta \psi} \end{aligned}$$

and

$$\begin{aligned} \mathrm{tr}(\nabla H\,\dot{L}) &= \mathrm{tr}(\nabla H\,(\dot\phi T\Delta^{-1}\psi + \phi T \Delta^{-1}\dot\psi)) \\ &= \mathrm{tr}(\dot\phi T\Delta^{-1}\psi\nabla H + \nabla H\phi T\Delta^{-1}\dot\psi) \\ &= \textstyle\sum_n \dot\phi\,(T\Delta^{-1}\psi\nabla H)_0 + \sum_n (\nabla H\phi T\Delta^{-1})_0\,\dot\psi \\ &= \textstyle\sum_n \dot\phi\,\frac{\delta H}{\delta\phi} + \sum_n \frac{\delta H}{\delta\psi}\,\dot\psi\,. \end{aligned}$$

Insertion of ∇H into (3.7) and (3.8) yields the Hamiltonian structures of (3.11):

$$\mathcal{P}_1(L)\nabla H = [\phi T\Delta^{-1}\psi, \nabla H_{\geq 0}]_{<1} = \phi T \Delta^{-1}\tfrac{\delta H}{\delta\phi} - \tfrac{\delta H}{\delta\psi} T\Delta^{-1}\psi\,,$$

and (after some computation)

$$\begin{aligned} \mathcal{P}_S(L)\nabla H \;\; &= \;\; \Big((\tfrac{\delta H}{\delta\psi})^{(1)} + \phi\psi\,\tfrac{\delta H}{\delta\psi} + \phi\rho(\psi\,\tfrac{\delta H}{\delta\psi}) - \phi\rho(\phi\,\tfrac{\delta H}{\delta\phi})\Big)T\Delta^{-1}\psi \\ &\quad -\phi T\Delta^{-1}\Big((\tfrac{\delta H}{\delta\phi})^{(-1)} + \psi\phi\,\tfrac{\delta H}{\delta\phi} - \psi\rho(\psi\,\tfrac{\delta H}{\delta\phi}) + \psi\rho(\psi\,\tfrac{\delta H}{\delta\psi})\Big)\,, \end{aligned}$$

i.e.,

$$\mathcal{P}_1(L)\ :\ \begin{pmatrix} \frac{\delta H}{\delta\phi} \\ \frac{\delta H}{\delta\psi} \end{pmatrix} \ \to\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta\phi} \\ \frac{\delta H}{\delta\psi} \end{pmatrix},$$

$$\mathcal{P}_S(L) : \begin{pmatrix} \frac{\delta H}{\delta\phi} \\ \frac{\delta H}{\delta\psi} \end{pmatrix} \ \to\ \begin{pmatrix} -\phi\,\rho\circ\phi & T+\phi\psi+\phi\,\rho\circ\psi \\ -T^{-1}-\psi\phi+\psi\,\rho\circ\phi & -\psi\,\rho\circ\psi \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta\phi} \\ \frac{\delta H}{\delta\psi} \end{pmatrix}$$

with

$$\rho = (T+1)(T-1)^{-1} :\ u(n)\ \to\ \sum_{j<n} u(j) - \sum_{j>n} u(j)\,.$$

The Hamiltonians of (3.11) are

$$H_1 = \mathrm{tr}(L) = \textstyle\sum_n \phi\psi \quad \text{and} \quad H_2 = \frac{1}{2}\,\mathrm{tr}(L^2) = \sum_n (\,\phi^{(1)}\psi + \frac{1}{2}\,(\phi\psi)^2\,)\,.$$

Remark 13: The connection with the version of the relativistic Toda lattice of Ref. [11] is as follows. A LU factorization (or, actually, a UL factorization) yields

$$L = T + \phi T\Delta^{-1}\psi = (d+T)(1-T^{-1}c)^{-1}$$

with $\psi = c\psi^{(1)}$ and $\phi\psi\psi^{(1)} = d\psi^{(1)} + \psi$, where we note

$$(1-T^{-1}c)^{-1} = (\,\psi^{-1}(1-T^{-1})\psi\,)^{-1} = \psi^{-1}T\Delta^{-1}\psi\,.$$

In Ref. [11] the adjoint of L was used, parametrized by the coordinates c and d. This explains why the underlying r-matrices for the (c,d) version of the relativistic Toda lattice equation and the usual Toda lattice equation (3.9) were slightly different, since Suris' adjoint of L is related to the class $\alpha = -1$ instead of $\alpha = 1$. In the present situation both the classical as well as the relativistic Toda model are representatives of the same abstract bi-Hamiltonian system $L_{t_1} = [L_{\geq 0}, L]$ restricted to different operator submanifolds.

3.3 The modified Lattice KP hierarchy and its reductions

We consider

$$\tilde{r}(\xi) = \xi_{\geq 1} - \xi_{<1} - 2\,(\xi\Delta^{-1})_0 \tag{3.12}$$

with adjoint

$$\tilde{r}^*(\xi) = \xi_{<0} - \xi_{\geq 0} - 2\,\Delta^{-1}\xi_0\,.$$

This r-matrix solves the modified Yang-Baxter equation (1.3) which follows easily from the following interpretation. One can rewrite the operators in g as being ordered w.r.t. powers of $\Delta = T-1$ instead of powers of T:

$$\begin{aligned} \xi &= u_N T^N + \cdots + u_1 T + u_0 + u_{-1}T^{-1} + \cdots \\ &= v_N \Delta^N + \cdots + v_1\Delta + v_0 + v_{-1}\Delta^{-1} + \cdots \end{aligned}$$

with

$$v_N = u_N\,, \quad v_{N-1} = u_{N-1} + N u_N\,, \quad \cdots \quad , \quad v_0 = u_0 + u_1 + \cdots + u_N\,, \quad \ldots .$$

There are two obvious decompositions of g into sub-(Lie)algebras. The first consists of powers ≥ 0 of Δ versus powers < 0. This is the same as splitting the operators into powers ≥ 0 and < 0 of T, so that the corresponding integrable systems are covered by the considerations of Section 3.2. The second decomposition $g = g_{\widetilde{\geq 1}} \oplus g_{\widetilde{<0}}$ into powers ≥ 1 of Δ versus powers < 1 is new. The r-matrix (3.12) is the identity on $g_{\widetilde{\geq 1}}$ and minus the identity on $g_{\widetilde{<1}}$, since $(\xi\Delta^{-1})_0 = u_N + \cdots + u_1 = v_0 - u_0$. We emphasize that we do not change the notation in this section: $\xi_{\geq 1}$, ξ_0, $\xi_{<1}$ etc. still denote the projection to various shift orders **w.r.t. powers of** T.

The linear Poisson tensor ($1.1.\mathcal{P}_1$) with the r-matrix (3.12)

$$\begin{aligned} \tilde{\mathcal{P}}_1(L)\nabla H &= \quad [\nabla H_{\geq 1} - (\nabla H \Delta^{-1})_0, L] - [\nabla H, L]_{\geq 0} - \Delta^{-1}[\nabla H, L]_0 \\ &= -[\nabla H_{<1} + (\nabla H \Delta^{-1})_0, L] + [\nabla H, L]_{<0} - \Delta^{-1}[\nabla H, L]_0 \end{aligned} \tag{3.13}$$

produces the integrable hierarchy [26]

$$\begin{aligned} L_{t_q} &= \quad [(L^q)_{\geq 1} - (L^q\Delta^{-1})_0, L] \\ &= -[(L^q)_{<1} + (L^q\Delta^{-1})_0, L] \end{aligned} \tag{3.14}$$

with the commuting Hamiltonians $H = \operatorname{tr}(L^{(q+1)})/(q+1)$. These equations are Miura related to the Lattice KP hierarchy (3.1, $\alpha = 1$) and shall be referred to as the modified Lattice KP hierarchy. For the Lax operators the link is given by a simple gauge transformation:

Theorem 14: *If* $L_{t_q} = [(L^q)_{\geq 0}, L]$ *and* $\Phi_{t_q} = ((L^q)_{\geq 0}\Phi)$, *then* $\tilde{L} = \Phi^{-1} L \circ \Phi$ *satisfies (3.14).*

Proof: One computes

$$\tilde{L}_{t_q} = [\,\Phi^{-1}(L^q)_{\geq 0} \circ \Phi - \Phi^{-1}\Phi_{t_q}\,,\, \tilde{L}\,] = [\,(\tilde{L}^q)_{\geq 1} - \Phi^{-1}((L^q)_{\geq 1}\Phi)\,,\, \tilde{L}\,]\,,$$

where

$$\Phi^{-1}((L^q)_{\geq 1}\Phi) = \Phi^{-1}((L^q)_{\geq 1} \circ \Phi\Delta^{-1})_0 = (\Phi^{-1}L^q \circ \Phi\Delta^{-1})_0 = (\tilde{L}^q\Delta^{-1})_0\,.$$

Q.E.D.

It is easily checked that the dynamics (3.14) can be restricted to Lax operators of the form

$$L = u_N T^N + u_{N-1}T^{N-1} + \cdots + u_M T^M + T^M\Delta^{-1}\Psi + \sum_{i=1}^{K} \phi_i T^M \Delta^{-1}\psi_i \tag{3.15}$$

or

$$L = u_N T^N + u_{N-1} T^{N-1} + \cdots + u_M T^M + T^M \Delta^{-1} \Psi \tag{3.16}$$

or

$$L = u_N T^N + u_{N-1} T^{N-1} + \cdots + u_M T^M \tag{3.17}$$

or [3]

$$L = u_N T^N + u_{N-1} T^{N-1} + \cdots + u_M T^M \,, \quad u_N + u_{N-1} + \cdots + u_M = 0\,. \tag{3.18}$$

Indeed, from the second representation of (3.14) it is clear that the highest shift order of the commutator coincides with the highest order of L. The invariance of the "low end" of the operators is verified with the first representation of (3.14) and (3.4). We note that the functions ϕ_i and ψ_i, Ψ, respectively, inherit the dynamics

$$\phi_{it_q} = ((L^q)_{\geq 1} - (L^q \Delta^{-1})_0)\phi_i)\,, \quad \psi_{it_q} = (-((L^q)_{\geq 1})^* + (L^q \Delta^{-1})_0)\psi_i)$$

of eigenfunctions and adjoint eigenfunctions, respectively. The equations associated with the Lax operators (3.17) are in general non-Hamiltonian. However, the operators (3.15) and (3.16) constitute integral manifolds of the linear Poisson tensor, so that the restriction to these operators is Hamiltonian:

Theorem 15: *The linear tensor (3.13) admits a proper restriction to Lax operators of the form (3.15) and (3.16) with $N \geq 1 \geq M$.*

This is verified by a straightforward computation using the two representations (3.13) and (3.4) to check the highest and lowest orders of $\tilde{\mathcal{P}}_1(L)\nabla H$.

We now turn to a discussion of the quadratic Poisson bracket associated with (3.14). As noted in Ref. [10], the skew-symmetric part of the r-matrix

$$\tfrac{1}{2}(\tilde{r} - \tilde{r}^*) : \ \xi \ \to \ \xi_{\geq 1} - \xi_{<0} - (\xi \Delta^{-1})_0 + \Delta^{-1}\xi_0$$

does *not* satisfy the modified Yang-Baxter equation, so that Semenov's quadratic tensor (1.1.$\mathcal{P}_2$) does *not* enjoy any Poisson properties. However, Suris' construction of Theorem 1 is capable of producing a suitable quadratic Poisson bracket:

Theorem 16: *Let* $\rho = \frac{T+1}{T-1} : \ u(n) \ \to \ \sum_{j<n} u(j) - \sum_{j>n} u(j)$ *. Then*

$$\begin{aligned}
A_1 &: \quad \xi \ \to \quad \xi_{\geq 1} - \xi_{<0} - 2(\xi\Delta^{-1})_0 - \rho(\xi_0) + 2\Delta^{-1}\xi_0\,,\\
A_2 &: \quad \xi \ \to \quad \xi_{\geq 1} - \xi_{<0} + \rho(\xi_0)\,,\\
S &: \quad \xi \ \to \quad \rho(\xi_0) - \xi_0 - 2\Delta^{-1}\xi_0\,,\\
S^* &: \quad \xi \ \to \ -\rho(\xi_0) - \xi_0 - 2(\xi\Delta^{-1})_0
\end{aligned}$$

[3] The constraint on the coefficients implies that such L may be written in the form

$$L = (v_{N-1}T^{N-1} + \cdots + v_M T^M)\,\Delta.$$

With $(L^q)_{\geq 1} - (L^q\Delta^{-1})_0 = (\cdots)\Delta$ it is clear that (3.18) is invariant under the flows (3.14).

satisfy all conditions of Theorem 1. The resulting Poisson tensor

$$\begin{aligned}\tilde{\mathcal{P}}_S(L)\nabla H \;=\;& (L\nabla H)_{\geq 1}L - L(\nabla HL)_{\geq 1} + \tfrac{1}{2}\,[L,\nabla H]_0 L + \tfrac{1}{2}\,L[L,\nabla H]_0 \\ & + \Delta^{-1}[L,\nabla H]_0 L + [L,(L\nabla H\Delta^{-1})_0] + \tfrac{1}{2}\,[\rho([\nabla H,L]_0),L] \\ =& -(L\nabla H)_{<0}L + L(\nabla HL)_{<0} + \tfrac{1}{2}\,[L,(L\nabla H+\nabla HL)_0] \\ & + \Delta^{-1}[L,\nabla H]_0 L + [L,(L\nabla H\Delta^{-1})_0] + \tfrac{1}{2}\,[\rho([\nabla H,L]_0),L]\end{aligned} \tag{3.19}$$

is compatible with the linear tensor (3.13) and admits a proper restriction to Lax operators of the form (3.15) and (3.16), $N \geq 1 \geq M$.

With $A_1 + S = A_2 + S^* = \tilde{r}$ one has to verify (1.4) in Theorem 2:

$$\begin{aligned} S\,(\,[A_2(\xi),\eta] + [\xi,A_2(\eta)]\,) \;&\overset{(i)}{=}\; 0 \;\overset{(ii)}{=}\; [\,S(\xi),S(\eta)\,]\,, \\ S^*(\,[A_1(\xi),\eta] + [\xi,A_1(\eta)]\,) \;&\overset{(iii)}{=}\; 0 \;\overset{(iv)}{=}\; [S^*(\xi),S^*(\eta)]\,. \end{aligned}$$

Here (i), (ii) and (iv) are (almost) trivial, whereas the verification of (iii) requires a substantial computational effort. We omit all details. The reduction properties of $\tilde{\mathcal{P}}_S$ are easily checked using the two representations (3.19) and (3.4).

Remark 17: After the exchange $A_1 \leftrightarrow A_2$ and $S \leftrightarrow S^*$ Theorem 1 yields the Poisson tensor

$$\begin{aligned}\tilde{\mathcal{P}}'_S(L)\nabla H \;=\;& (L\nabla H)_{\geq 1}L - L(\nabla HL)_{\geq 1} + \tfrac{1}{2}\,[L,\nabla H]_0 L + \tfrac{1}{2}\,L[L,\nabla H]_0 \\ & + L\Delta^{-1}[L,\nabla H]_0 + [L,(\nabla HL\Delta^{-1})_0] - \tfrac{1}{2}\,[\rho([\nabla H,L]_0),L] \\ =& -(L\nabla H)_{<0}L + L(\nabla HL)_{<0} + \tfrac{1}{2}\,[L,(L\nabla H+\nabla HL)_0] \\ & + L\Delta^{-1}[L,\nabla H]_0 + [L,(\nabla HL\Delta^{-1})_0] - \tfrac{1}{2}\,[\rho([\nabla H,L]_0),L]\end{aligned} \tag{3.20}$$

which is again compatible with the linear tensor (3.13) and admits a proper restriction to Lax operators of the form (3.18). This yields *the* Hamiltonian formulation of the integrable hierarchies arising from Lax operators (3.18). We note that the linear tensor (3.13) cannot be restricted properly to such operators. Dirac reduction results in a nonlocal bracket stemming from $\tilde{\mathcal{P}}_1$.

Example 18: The gauge transformation $L \to \tilde{L} = \phi^{-1}\,L \circ \phi$ of Theorem 14 applied to the Lax operator of Example 12 yields

$$L = T + \phi T\Delta^{-1}\psi \quad \to \quad \tilde{L} = u\,T + T\Delta^{-1}\Psi$$

with

$$u = \phi^{-1}\phi^{(1)}\,,\ \Psi = \psi\phi\,. \tag{3.21}$$

The basic equation of the integrable hierarchy (3.14) for $\tilde{L}$ is the relativistic Toda lattice equation [27, 28]

$$u_{t_1} = u\,(\,u^{(1)} - u + \Psi^{(1)} - \Psi\,)\,, \quad \Psi_{t_1} = u\Psi - u^{(-1)}\Psi^{(-1)} \tag{3.22}$$

which is related to (3.11) by the "Miura transformation" (3.21). The first two Hamiltonian structures found in Ref. [28] can be retrieved from (3.13) and (3.19) by inserting $\nabla H = T^{-1}\frac{\delta H}{\delta u} + \frac{\delta H}{\delta \Psi}$. From

$$\tilde{\mathcal{P}}_1(\tilde{L})\nabla H = u\,(\,(\tfrac{\delta H}{\delta \Psi})^{(1)} - \tfrac{\delta H}{\delta \Psi}\,)\,T + T\Delta^{-1}\,(\,u\,\tfrac{\delta H}{\delta u} - (u\,\tfrac{\delta H}{\delta u})^{(-1)}\,)$$

and a corresponding expression for $\tilde{\mathcal{P}}_S(L)\nabla H$ one obtains

$$\tilde{\mathcal{P}}_1(L)\ :\ \begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta \Psi} \end{pmatrix} \ \to\ \begin{pmatrix} 0 & u\,(T-1) \\ (1-T^{-1})\,u & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta \Psi} \end{pmatrix},$$

$$\tilde{\mathcal{P}}_S(L) :\ \begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta \Psi} \end{pmatrix} \ \to\ \begin{pmatrix} u\,(T-T^{-1})\,u & u\,(T-1)\,(u\,T+\Psi) \\ (T^{-1}u+\Psi)\,(1-T^{-1})\,u & u\Psi\,T - T^{-1}u\Psi \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta u} \end{pmatrix}.$$

The Hamiltonians of (3.22) are

$$H_1 = \mathrm{tr}(L) = \textstyle\sum_n \Psi \quad \text{and} \quad H_2 = \frac{1}{2}\,\mathrm{tr}(L^2) = \sum_n \left(u\Psi + \frac{1}{2}\,\Psi^2\right)\ .$$

Example 19: We discuss an example of the class (3.18). The basic equation associated with the Lax operator $L = (u + v\,T^{-1})\,\Delta$ is

$$u_{t_1} = u\,(v^{(1)} - v)\,, \quad v_{t_1} = v\,(u^{(-1)} - u)\,.$$

Insertion of $\nabla H = \Delta^{-1}(\frac{\delta H}{\delta u} + T\,\frac{\delta H}{\delta v})$ into (3.20) yields the Poisson structure of this equation:

$$\mathcal{P}'_S(L)\nabla H = \Big(u\ \big(\,(v\,\tfrac{\delta H}{\delta v})^{(1)} - v\,\tfrac{\delta H}{\delta v}\,\big) + v\ \big(\,u\,\tfrac{\delta H}{\delta u} - (u\,\tfrac{\delta H}{\delta u})^{(-1)}\,\big)\ T^{-1}\Big)\,\Delta\,,$$

i.e.,

$$\tilde{\mathcal{P}}'_S(L)\ :\ \begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta v} \end{pmatrix} \ \to\ \begin{pmatrix} 0 & u\,(T-1)\,v \\ v\,(1-T^{-1})\,u & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta v} \end{pmatrix}\,.$$

The Hamiltonian is $H = \mathrm{tr}(L) = \sum_n (v - u)$.

4 Conclusions

We have derived the Poisson brackets associated with scalar Lax equations $L_t = [A, L]$, where L is a shift operator and A involves only shifts into one lattice direction. The main examples of such systems are the Lattice KP hierarchy and the modified Lattice KP hierarchy, in which A is generated via (fractional) powers of L. They include the Toda model and the relativistic Toda model as the most prominent equations. As already observed by Suris, both systems may be regarded as the same dynamical system (the Lattice KP hierarchy) restricted to different operator submanifolds. The unifying structure is provided by simple r-matrices on the algebra of shift symbols. By restriction or reduction to finite

Lax operators they generate Poisson brackets which yield the multi-Hamiltonian formulations for a variety of integrable lattice hierarchies. Reductions of the Lattice KP hierarchy include all Lax operators of the form

$$L = T^N + u_{N-1}T^{N-1} + \cdots + u_M T^M + \sum_{i=1}^{K} \phi_i T^M \Delta^{-1} \psi_i$$

with arbitrary K and $N \geq 1 \geq M$. Reductions of the modified Lattice KP include the Lax operators

$$L = u_N T^N + \cdots + u_M T^M + T^M \Delta^{-1} \Psi + \sum_{i=1}^{K} \phi_i T^M \Delta^{-1} \psi_i$$

with arbitrary K and $N \geq 1 \geq M$.

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