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Darboux Theorems and Wronskian Formulas for Integrable Systems I: Constrained KP Flows

W. Oevel
Dept. Mathematical Sciences
Loughborough University of Technology
Loughborough LEII 3TU
U.K.

Abstract

Generalizations of the classical Darboux theorem are established for pseudo- differential scattering operators of the form $L = \sum_{i=0}^{N} u_i \partial^i + \sum_{i=1}^{m} \Phi_i \partial^{-1} \Psi_i^{\dagger}$. Iteration of the Darboux transformation leads to a gauge transformed operator with coefficients given by Wronskian formulas involving a set of eigenfunctions of L. Nonlinear integrable partial differential equations are associated with the scattering operator L which arise as a symmetry reduction of the multicomponent KP hierarchy. With a suitable linear time evolution for the eigenfunctions the Darboux transformation is used to obtain solutions of the integrable equations in terms of Wronskian determinants.

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1 Introduction

Since the mid 1970's Darboux transformations (DTs) have been shown to be a powerful tool for studying the nonlinear integrable systems of soliton theory. Darboux's original result [1] is a transformation of the Schrödinger equation $\Phi_{xx} + u\Phi = \lambda\Phi$ with different potentials. Given a first eigenfunction for a potential u, it can be used to map all other eigenfunctions to new eigenfunctions for a Schrödinger operator with a modified potential. Crum [2] considered the iteration of the DT, when a collection of eigenfunctions is given. The iterated transformation can be described in a compact way involving the Wronskian determinant of the eigenfunctions. The key fact leading to the spectacular properties of soliton systems is their connection to linear scattering problems. For the celebrated Korteweg-de Vries (KdV) equation the associated linear problem is the eigenvalue equation for the Schrödinger operator. Hence, it is not surprising that Darboux's result has implications for the KdV. In particular, it was shown by Crum that the DT adds an eigenvalue to the spectrum of the Schrödinger operator. On the level of the nonlinear equation this means that solitons are added by this transformation ([3]), and Crum's Wronskian formulas for the iterated DT correspond to compact Wronskian representations of the n-soliton solutions of the KdV ([4]).

Generalizations of these ideas for other spectral equations and their associated soliton systems have been successfully applied to a variety of systems. This includes spectral problems for ordinary differential operators, which are associated with soliton systems in 1+1-dimensions, as well as scattering problems in several variables and their associated nonlinear equations in higher dimensions. In particular, interesting localized coherent structures (dromions) may be constructed using an appropriate Darboux formalism ([5]). An excellent overview summarizing various forms of (generalized) DTs and their applications in soliton theory is the recent book [6] by Matveev and Salle.

The purpose of the present paper is the following. First, we review the concept of DTs in terms of pseudo-differential symbols. The formal integration symbols provide a very efficient tool to characterize the DT as a simple gauge transformation of the Lax pair. In ([7, 8]) this terminology was used to give a unified description of DTs as well as Miura transformations and reciprocal links in terms of pseudo-differential gauge operators. In this formulation the transformation does not depend on the special nature of the Lax pair involved. In particular, not only differential Lax operators but also pseudo-differential scattering problems may be included. Our final aim is to apply these results to Lax operators of the form

$$L = u_N\partial^N + u_{N-1}\partial^{N-1}.. + u_0 + \sum_{i=1}^{m} \Phi_i\partial^{-1}\Psi_i^\dagger \tag{1.1}$$

and obtain Wronskian type solution formulas for the associated nonlinear evolution equations.

Scattering operators of the class (1.1) were encountered recently in the context of reductions of the KP hierarchy, which generalizes a reduction concept originally used to reduce soliton systems in 1+1 dimensions to finite dimensional equations. Typical reduction schemes from partial to ordinary differential equations involve pole expansions [9, 10] or stationary flows [11] and reductions to pure soliton submanifolds [12]. For the latter case a systematic "nonlinearization" procedure was proposed by Cao [13]. The main idea is that squared eigenfunctions (i.e. the products of eigenfunctions Φ_i

and adjoint eigenfunctions Ψ_i) associated with the underlying linear scattering problems may be regarded as conserved covariants of the nonlinear hierarchy. Finite dimensional invariant submanifolds for these flows are given by the stationary points of conserved covariants, hence constraints such as $u = \sum_i \Phi_i \Psi_i$, say, will be preserved under the dynamics, if the potential u is a conserved covariant of the nonlinear systems. This idea has been generalized to reduce soliton equations in 2+1 dimensions to simpler integrable partial differential equations in 1+1 dimensions. In particular, for the KP hierarchy the space-derivative $(\sum_i \Phi_i \Psi_i)_x$ of squared eigenfunctions represents a symmetry generator ([14, 15]). Hence, using any symmetry u_{t_N} of the KP-hierarchy the constraint $u_{t_N} = (\sum_i \Phi_i \Psi_i)_x$ is compatible with all 2+1-dimensional flows, thus leading to a hierarchy of commuting integrable equations in 1+1-dimensions ([16-22]). These reductions were found to have a simple description using Sato's construction ([23-27]) of the (scalar) KP hierarchy. It was observed ([28-31]) that the symmetry constraint $u_{t_N} = (\Phi\Psi)_x$ of the nonlinearization approach yields equations associated with scalar Lax operators of the form (1.1). A general description of these systems and their bi-Hamiltonian nature is given in [32], where the pseudo-differential part is restricted to one pair of eigenfunctions and adjoint eigenfunctions ($m = 1$) for simplicity. In the present paper we generalize this reduction concept to the multicomponent case and identify the symmetry generated by squared eigenfunctions in a systematic way. It turns out that squared eigenfunction are naturally connected with the pseudo-differential operator $\sum_i \Phi_i \partial^{-1} \Psi_i^\dagger$. As a result, operators of the form (1.1) arise in the corresponding reduction of the multicomponent KP hierarchy.

In Section 2 we briefly review the essentials of pseudo-differential symbols to be used for the description of the (constrained) KP hierarchy. We review the fact that for each pair of eigenfunctions Φ and adjoint eigenfunctions Ψ the squared eigenfunction can be integrated to a potential $S = S(\Psi, \Phi)$ satisfying $S_x = \Psi^\dagger \Phi$ and a corresponding time evolution. These potentials will play a crucial role in the DT of the adjoint eigenfunctions. In particular, with these potentials a proper spectral problem can be formulated for pseudo-differential operators of the form (1.1).

In Section 3 a general formulation for the DT for Lax pairs is presented in terms of pseudo-differential symbols. For Lax pairs consisting of differential operators the terminology of formal integration symbols is not really necessary to obtain these results. However, also in this case the pseudo-differential formalism provides a very efficient technical tool to phrase the DT as a simple gauge transformation of the Lax pair. In this formulation the results also apply to general pseudo-differential Lax operators, so that a unified description of the DT is achieved including operators of the form (1.1). Iterations of the DT generated by a collection of eigenfunctions are considered. The resulting Lax pairs are characterized by a gauge operator, which involves the Wronskian matrix of these eigenfunctions.

In Section 4 we review the construction of a large class of integrable systems through Sato's construction of the (multicomponent) KP hierarchy, where the DT can be described as a simple transformation of a dressing operator. For the Lax pairs derived from the dressing operator this transformation provides the DT of Section 3. In this case a convenient parametrization of the Lax pairs in terms of the dressing operator is given. As a result, the DT does not only leave the dynamics of the Lax pairs invariant, but also the parametrization, i.e. the form of the operators, is preserved.

In Section 5 we generalize the concept of "N-reduction" in Sato's theory which originally reduces the

KP hierarchy to the nonlinear equations associated with differential scattering problems of Gelfand-Dikii type (i.e. (1.1) with no pseudo-differential part). For this generalization we review that the KP hierarchy admits a one parameter symmetry group generated by squared eigenfunctions. The formulation given here applies to the general multicomponent case and reduces to the well known results for the special case of the scalar KP hierarchy. A symmetry reduction involving this symmetry reduces the Lax operators of Sato's approach to operators of the form (1.1).

In Section 6 we show that the DT will leave the class of operators (1.1) invariant, if the triggering eigenfunctions are solutions of a spectral problem involving the potential associated with eigenfunctions and adjoint eigenfunctions. This, in combination with the results of Section 3 on the temporal aspects of the DTs, provides an efficient tool to generate exact solutions (solitons) of the nonlinear equations associated with scattering operators of the form (1.1).

In Section 7 the general results are illustrated for the two-component KP hierarchy. We review the construction of the 2+1-dimensional AKNS system, which is the simplest realization of the nonlinear systems residing in the two-component KP hierarchy. The constraints of Section 5 leads to a reduction of these equations to a "nonlinearized" Nonlinear Schrödinger System (NLS). Solution formulas are obtained from application of the general Darboux concept to this special case.

2 General background and basic definitions

We consider pseudo-differential symbols of the form

$$L \in g = \{ \sum_{i << \infty} u_i \, \partial^i \quad ; \quad u_i \in g_0 \} \quad , \tag{2.1}$$

where g_0 is a finite dimensional matrix algebra. The coefficients u_i are functions of a "space variable" x. In the following we will always identify functions with the multiplication operator given by this function. The space g of all operators (2.1) is endowed with an algebra structure requiring that for positive n the symbol ∂^n is the power of the differential operator $\partial = \partial/\partial x$, whereas negative powers are formal integrations defined by the multiplication rule

$$\partial^n u = \sum_{k \geq 0}^{\infty} \begin{pmatrix} n \\ k \end{pmatrix} u^{(k)} \, \partial^{n-k} \quad . \tag{2.2}$$

Here $u^{(k)}$ is the kth derivative of u w.r.t. x. This, in particular, implies the following useful rules

$$\partial^{-1} u = u \partial^{-1} - u_x \partial^{-2} + u_{xx} \partial^{-3} \mp \ldots \quad , \quad u \partial^{-1} = \partial^{-1} u + \partial^{-2} u_x + \partial^{-3} u_{xx} + \ldots \quad . \tag{2.3}$$

We will use the following –fairly obvious– notation for projections of a pseudo-differential symbol $L = \sum_i u_i \partial^i$ to various differential orders:

$$P_{\geq k}(L) = \sum_{i \geq k} u_i \partial^i \quad , \quad P_{<k}(L) = \sum_{i<k} u_i \partial^i \quad , \quad P_k(L) = u_k \partial^k \quad . \tag{2.4}$$

Warning: The projection P_0 to the zero order term will play an important role in the following. If an operator L and a function Φ is given, we will always use the notation $L\Phi$ for the operator which is the composition of L with the multiplication operator given by Φ. In order to avoid confusion in our notation we will never apply a differential operator to a function. Instead, we observe that the composition of $L = u_0 + u_1\partial + ... + u_N\partial^N$ with the multiplication by Φ is of the form

$$L\Phi = (u_0+u_1\partial+..+u_N\partial^N)\Phi = (u_0\Phi+u_1\Phi_x+..+u_N\Phi^{(N)})+(u_1\Phi+2u_2\Phi_x+..)\partial+...+(..)\partial^N \ , \quad (2.5)$$

so that application of the operator L to the function Φ can be described as the zero order term $P_0(L\Phi)$ of the operator $L\Phi$. Hence, for arbitrary pseudo-differential operator L the symbol $P_0(L\Phi)$ may be understood as the function obtained from letting the differential part $P_{\geq 0}(L)$ act on the function Φ.

A further important projection is the residue of a pseudo-differential symbol given by

$$res(\sum_i u_i\partial^i) = P_{-1}(\sum_i u_i\partial^i)\partial = u_{-1} \quad . \qquad (2.6)$$

The relevance of this residue for our considerations is given by the following observation. Let $\dagger$ denote the matrix transposition of the coefficients $u_i \in g_0$. With the symbol $*$ we introduce the usual transposition

$$(\sum_i \ u_i\partial^i \ )^* = \sum_i (-1)^i\partial^i u_i^\dagger \qquad (2.7)$$

of pseudo-differential symbols. We have the property that for any matrix differential operator M and any pairs of vector or matrix functions Φ, Ψ the bilinear expression

$$\Psi^\dagger P_0(M\Phi) - P_0(M^*\Psi)^\dagger\Phi \qquad (2.8)$$

is a perfect x-derivative of some bilinear form $A(\Psi,\Phi)$. With the definition of the residue this form can be expressed in a compact way as

$$A(\Psi,\Phi) = res(\partial^{-1}\Psi^\dagger M\Phi\partial^{-1}) \ . \qquad (2.9)$$

The pseudo-differential symbol ∂^{-1} used in the following considerations should be regarded as a very effective technical tool to simplify computations involving compositions of differential operators. We list some useful identities

$$\begin{array}{lclllcl} P_{\geq 0}(\alpha^*) & = & P_{\geq 0}(\alpha)^* \ , & & P_{<0}(\alpha^*) & = & P_{<0}(\alpha)^* \ , \\ res(\alpha) & = & -res(\alpha^*)^\dagger \ , & & res(\alpha)_x & = & res(\partial\alpha - \alpha\partial) \ , \\ res(\alpha\partial^{-1}) & = & P_0(\alpha) \ , & & res(\partial^{-1}\alpha) & = & P_0(\alpha^*)^\dagger \ , \\ res(\Phi\ \alpha\ \Psi^\dagger) & = & \Phi\ res(\alpha)\ \Psi^\dagger \ , & & P_{<0}(\alpha\partial^{-1}) & = & P_0(\alpha)\partial^{-1} + P_{<0}(\alpha)\partial^{-1} \ , \\ P_{<0}(\partial^{-1}\alpha) & = & \partial^{-1}P_0(\alpha^*)^\dagger + \partial^{-1}P_{<0}(\alpha) \ , & & & & \end{array} \qquad (2.10)$$

which hold for arbitrary pseudo-differential operators $\alpha \in g$ and arbitrary matrix or vector functions Φ, Ψ. These identities are readily verified using the definitions provided. As a consequence, we derive the following more complicated identity, which holds for all differential operators $\alpha = P_{\geq 0}(\alpha), \beta =$

$P_{\geq 0}(\beta) \in g$:

$$\begin{aligned} res(\partial^{-1}\alpha\beta\partial^{-1}) &= res(\partial^{-1}\alpha P_{\geq 1}(\beta)\partial^{-1}) + res(\partial^{-1}\alpha P_0(\beta)\partial^{-1}) \\ &= res(P_{<0}(\partial^{-1}\alpha)P_{\geq 1}(\beta)\partial^{-1}) + res(\partial^{-1}\alpha P_0(\beta)\partial^{-1}) \\ &= res(\partial^{-1}P_0(\alpha^*)^\dagger P_{\geq 1}(\beta)\partial^{-1}) + res(\partial^{-1}\alpha P_0(\beta)\partial^{-1}) \\ &= res(\partial^{-1}P_0(\alpha^*)^\dagger \beta\partial^{-1}) + res(\partial^{-1}\alpha P_0(\beta)\partial^{-1}) \ . \end{aligned} \tag{2.11}$$

For a given differential operator M we will consider eigenfunctions and adjoint eigenfunctions, which are defined as solutions of the evolution equations $\Phi_t = P_0(M\Phi)$ and $\Psi_t = -P_0(M^*\Psi)$. We first establish the important fact that associated with each pair Φ, Ψ there is a potential integrating the "squared eigenfunction" $\Psi^\dagger\Phi$.

Lemma 1:

a) For any matrix differential operator M and any pair of solutions Φ and Ψ of $\Phi_t = P_0(M\Phi)$, $\Psi_t = -P_0(M^*\Psi)$ there exists a potential $S(\Psi,\Phi)$ satisfying

$$S(\Psi,\Phi)_x = \Psi^\dagger\Phi \quad , \quad S(\Psi,\Phi)_t = res(\partial^{-1}\Psi^\dagger M\Phi\partial^{-1}) \ . \tag{2.12}$$

b) Let $\mathcal{A}$ be a set of indices, let $\{ \ M_A \ , \ A \in \mathcal{A} \ \}$ be a collection of matrix-valued differential operators, which are functions of a collection of independent "time" variables $\{ \ t_A \ , \ A \in \mathcal{A} \ \}$. We assume that the zero-curvature condition

$$M_{Bt_A} - M_{At_B} = [M_A, M_B] \tag{2.13}$$

is satisfied for all pairs of indices $A, B \in \mathcal{A}$. Then, if Φ and Ψ solve the compatible linear evolution equations $\Phi_{t_A} = P_0(M_A\Phi)$ and $\Psi_{t_A} = -P_0(M_A^*\Psi)$, there exists a potential $S(\Psi,\Phi)$ satisfying

$$S(\Psi,\Phi)_x = \Psi^\dagger\Phi \quad , \quad S(\Psi,\Phi)_{t_A} = res(\partial^{-1}\Psi^\dagger M_A\Phi\partial^{-1}) \ . \tag{2.14}$$

We remark that the zero-curvature condition (2.13) guarantees that all the linear equations are compatible and can be satisfied simultaneously. Using the identities (2.10) the existence of this potential is readily checked, as

$$\begin{aligned} (S_{t_A})_x = res(\Psi^\dagger M_A\Phi\partial^{-1} - \partial^{-1}\Psi^\dagger M_A\Phi) &= \Psi^\dagger P_0(M_A\Phi) - P_0(M_A^*\Psi)^\dagger\Phi \\ &= \Psi^\dagger\Phi_{t_A} + \Psi^\dagger_{t_A}\Phi = (S_x)_{t_A} \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} (S_{t_A})_{t_B} - (S_{t_B})_{t_A} &= -res(\partial^{-1}P_0(M_B^*\Psi)^\dagger M_A\Phi\partial^{-1}) + res(\partial^{-1}\Psi^\dagger M_A P_0(M_B\Phi)\partial^{-1}) \\ &\quad + res(\partial^{-1}P_0(M_A^*\Psi)^\dagger M_B\Phi\partial^{-1}) - res(\partial^{-1}\Psi^\dagger M_B P_0(M_A\Phi)\partial^{-1}) \\ &\quad + res(\partial^{-1}\Psi^\dagger \ (M_{At_B} - M_{Bt_A}) \ \Phi\partial^{-1}) \\ &= res(\partial^{-1}\Psi^\dagger \ (\ [M_A, M_B] + M_{At_B} - M_{Bt_A} \ ) \ \Phi\partial^{-1}) = 0 \ , \end{aligned} \tag{2.16}$$

where we have used the operator identity (2.11). Case a) follows with $M_A = M$, $t_A = t$, $M_B = \partial$ and $t_B = x$.

We remark that we may consider vector-valued solutions Φ and Ψ of the linear problems, where both Φ and Ψ are given as columns. In this case the corresponding potential S is scalar-valued. Alternatively, vector solutions may be combined to matrix-valued solutions, so that a pair of matrices Φ and Ψ gives rise to a matrix potential $S(\Psi, \Phi)$. In any case we regard this potential as a bi-linear form of its two entries, which may be thought of as a contour integral in the space spanned by x and the time variables t_A. We note that the space variable x may be regarded as the special time variable associated with the matrix operator $M = \partial$.

The potential $S(\Psi, \Phi)$ associated with pairs of functions Φ, Ψ will play an important role in the DT of operators of the form (1.1), which occur naturally in the reduction of the KP hierarchy. Again, in (1.1) the fields Φ_i and Ψ_i may either be matrices or vectors (columns). In the latter case an expression such as $\Phi_i \partial^{-1} \Psi_i^\dagger$ is to denote the matrix

$$(\Phi_i \partial^{-1} \Psi_i^\dagger)_{ab} = (\Phi_i)_a \partial^{-1} (\Psi_i)_b \quad . \tag{2.17}$$

We will consider Lax equations

$$L_t = [M, L] \quad , \tag{2.18}$$

where M is some suitable differential operator. We will give a systematic construction of such Lax equations in Section 4, where both L and suitable M's are obtained from the KP hierarchy.

We remark that the Lax equation (2.18) will determine the nonlinear dynamical equations for the fields $u_N, .., u_0$ in (1.1) uniquely, as the time evolution of the field u_j, say, is simply given as the coefficient of the jth differential order of the right hand side $[M, L]$. However, certain ambiguities in the dynamics for the fields Φ_i, Ψ_i will remain. This, of course, is to be expected, as e.g. the replacement of a pair (Φ_i, Ψ_i) by $(\Phi_i f(t), f^{-1}(t) \Psi_i)$ will result in the same L, so that the dynamics of L cannot bear the complete information on the dynamics for the fields Φ_i and Ψ_i. However, we may assume that these fields obey the dynamics of eigenfunctions and adjoint eigenfunctions associated with the dynamics (2.18).

Lemma 2:

Let L be of the form (1.1). The Lax equation (2.18) for L is consistent with the evolutions

$$\Phi_{it} = P_0(M\Phi_i) \quad , \quad \Psi_{it} = -P_0(M^* \Psi_i) \quad , \quad i = 1, .., m \quad . \tag{2.19}$$

To see this we write $L = l + \sum_i \Phi_i \partial^{-1} \Psi_i^\dagger$, where $l = u_N \partial^N + .. + u_0$ is the differential part of L. We may extract the positive and negative differential orders of (2.18) and find

$$\begin{aligned} l_t &= P_{\geq 0}([M, L]) = [M, l] + \textstyle\sum_i P_{\geq 0} \left(\ M\Phi_i \partial^{-1} \Psi_i^\dagger - \Phi_i \partial^{-1} \Psi_i^\dagger M \ \right) \quad , \\ \textstyle\sum_i \left(\ \Phi_{it} \partial^{-1} \Psi_i^\dagger + \Phi_i \partial^{-1} \Psi_{it}^\dagger \ \right) &= P_{\leq 0}([M, L]) = \textstyle\sum_i \left(\ P_0(M\Phi_i) \partial^{-1} \Psi_i^\dagger - \Phi_i \partial^{-1} P_0(M^* \Psi_i)^\dagger \ \right) \quad . \end{aligned} \tag{2.20}$$

Hence, if we assume the evolutions (2.19) of eigenfunctions and adjoint eigenfunctions, then the evolution (2.20.ii) of the negative part of L is satisfied identically. The evolution of the fields $u_N,..,u_0$ parametrizing the positive part of L is obtained from (2.20.i).

For a differential operator L the Lax equation (2.18) may be thought of as the compatibility condition for the eigenvalue problem $P_0(L\Phi) = \lambda\Phi$ and the time evolution $\Phi_t = P_0(M\Phi)$ for the eigenfunctions. A pseudo-differential operator L, however, is supposed to act on a suitable set of test functions, so that it is not proper to impose an eigenvalue problem for L in terms of a function Φ usually assumed to live in some Hilbert space. In order to overcome this technicality we will replace the formal eigenvalue problem for (1.1) by a proper ordinary differential eigenvalue problem with the following Lemma.

Lemma 3:

Let L of the form (1.1) satisfy $L_t = [M, L]$, $\Phi_{it} = P_0(M\Phi_i)$, $\Psi_{it} = -P_0(M^*\Psi_i)$ with some differential operator M. Then the spectral problem

$$u_N\Phi^{(N)} + .. + u_1\Phi_x + u_0\Phi + \sum_{i=1}^{m} \Phi_i S_i = \Phi\,\Lambda \quad , \quad S_{ix} = \Psi_i^\dagger\Phi \quad , \quad i = 1, .., m \quad , \tag{2.21}$$

with a constant “spectral” matrix Λ is compatible with the evolution

$$\Phi_t = P_0(M\Phi) \quad , \quad S_{it} = res(\partial^{-1}\Psi_i^\dagger M\Phi\partial^{-1}) \quad . \tag{2.22}$$

For the **proof** we observe that we may introduce the potentials $S_i = S(\Psi_i, \Phi)$ according to Lemma 1. Hence it remains to show that

$$\frac{d}{dt}(\ P_0(L\Phi) + \sum_i \Phi_i S_i - \Phi\Lambda\ ) = 0 \quad , \tag{2.23}$$

where $P_0(L\Phi) = P_0(l\Phi) = u_N\Phi^{(N)} + .. + u_0\Phi$ is given by the differential part $l = P_{\geq 0}(L)$ of L. Inserting (2.20.i) and the dynamics of Φ, Φ_i, S_i this expression is given by

$$\begin{aligned} &P_0(l_t\Phi + l\Phi_t) + \textstyle\sum_i(\Phi_{it}S_i + \Phi_i S_{it}) - \Phi_t\Lambda = \\ &\quad P_0(Ml\Phi - lM\Phi + \textstyle\sum_i(P_{\geq 0}(M\Phi_i\partial^{-1}\Psi_i^\dagger)\Phi - P_{\geq 0}(\Phi_i\partial^{-1}\Psi_i^\dagger M)\Phi) + lM\Phi) \\ &\qquad + \textstyle\sum_i P_0(M\Phi_i)S_i + \sum_i res(\Phi_i\partial^{-1}\Psi_i^\dagger M\Phi\partial^{-1}) - P_0(M\Phi)\Lambda \quad . \end{aligned} \tag{2.24}$$

We observe

$$\begin{aligned} P_0(P_{\geq 0}(M\Phi_i\partial^{-1}\Psi_i^\dagger)\Phi) + P_0(M\Phi_i)S_i &= P_0(P_{\geq 1}(M\Phi_i)\partial^{-1}\Psi_i^\dagger\Phi) + P_0(M\Phi_i)S_i \\ &= P_0(P_{\geq 1}(M\Phi_i)\partial^{-1}\partial S_i) + P_0(M\Phi_i)S_i \\ &= P_0(P_{\geq 0}(M\Phi_i)S_i) = P_0(M\Phi_i S_i) \quad , \end{aligned} \tag{2.25}$$

and

$$P_0(P_{\geq 0}(\Phi_i\partial^{-1}\Psi_i^\dagger M)\Phi) = res(P_{\geq 0}(\Phi_i\partial^{-1}\Psi_i^\dagger M\Phi)\partial^{-1}) = res(\Phi_i\partial^{-1}\Psi_i^\dagger M\Phi\partial^{-1}) \ , \tag{2.26}$$

so that (2.24) becomes

$$P_0(M(l\Phi + \sum_i \Phi_i S_i - \Phi\Lambda)) = P_0(M(P_0(l\Phi) + \sum_i \Phi_i S_i - \Phi\Lambda)) = 0 \quad . \tag{2.27}$$

Hence the spectral problem (2.21) is invariant under the flow (2.22).

3 A general Darboux theorem for Lax pairs

We consider nonlinear dynamical systems which are given by a Lax equation $L_t = [M, L]$ with a pseudo-differential operator L and a differential operator M. We consider a gauge transformation of the Lax pair L, M, which is given by an invertible (differential or pseudo-differential) transformation operator T via

$$L \quad \to \quad \tilde{L} = TLT^{-1} \quad , \quad M \quad \to \quad \tilde{M} = TMT^{-1} + T_t T^{-1} \quad . \tag{3.1}$$

This form of the transformation automatically guarantees that the new pair $\tilde{L}, \tilde{M}$ again satisfies the Lax equation $\tilde{L}_t = [\tilde{M}, \tilde{L}]$. In order to ensure that the gauge transformed operator $\tilde{M}$ is again a differential operator, we assume that the term $T_t T^{-1}$ in $\tilde{M}$ cancels the negative differential orders which might occur in TMT^{-1}. Hence, we assume that the transformation operator T satisfies the nonlinear evolution equation

$$T_t = -P_{<0}(TMT^{-1})T \quad . \tag{3.2}$$

We remark that this is a somewhat restrictive assumption, as the necessary condition to ensure that $\tilde{M}$ is differential only given by $P_{<0}(T_t T^{-1}) = -P_{<0}(TMT^{-1})$. With (3.2) the transformations to be considered are of the form

$$L \quad \to \quad \tilde{L} = TLT^{-1} \quad , \quad M \quad \to \quad \tilde{M} = P_{\geq 0}(TMT^{-1}) \quad . \tag{3.3}$$

Lateron, in the context of the KP hierarchy, we will consider a fixed Lax operator L for which there is a hierarchy of Lax equations $L_{t_A} = [M_A, L]$, where A is an index from some index set $\mathcal{A}$. To each operator M_A a different flow parameter ("time") t_A is associated. We assume that for each pair of indices $A, B \in \mathcal{A}$ the differential operators satisfy the zero-curvature condition (2.13), which may be regarded as the compatibility condition of the linear problems $\Phi_{t_A} = P_0(M_A\Phi)$ and $\Phi_{t_B} = P_0(M_B\Phi)$. If the gauge transformation $M_A \to \tilde{M}_A = TM_AT^{-1} + T_{t_A}T^{-1}$ is applied to each operator, then the identity

$$\tilde{M}_{At_B} - \tilde{M}_{Bt_A} + [\tilde{M}_A, \tilde{M}_B] \; = \; T(M_{At_B} - M_{Bt_A} + [M_A, M_B])T^{-1} \tag{3.4}$$

guarantees that the zero-curvature condition (2.13) is inherited to each new pair $\tilde{M}_A, \tilde{M}_B$ for any choice of the transformation operator T. If we assume that the gauge operator T satisfies the equations

$$T_{t_A} = -P_{<0}(TM_AT^{-1})T \tag{3.5}$$

for any $A \in \mathcal{A}$, then all the gauge transformed operators $\tilde{M}_A$ are again differential operators given by $\tilde{M}_A = P_{\geq 0}(TM_AT^{-1})$. We note that one may indeed consider (3.5) simultaneously for all $A \in \mathcal{A}$, as with the identity

$$(T_{t_B})_{t_A} - (T_{t_A})_{t_B} = P_{<0}(\; T \; (M_{At_B} - M_{Bt_A} + [M_A, M_B]) \; T^{-1} \;) \; T \tag{3.6}$$

all equations (3.5) are compatible. In the following we will drop the index labeling the elements of the hierarchy, as for each flow the gauge transformations can be treated separately.

The essential observation behind the DTs to be discussed in this section is the fact that the <u>nonlinear</u> equation (3.2) for the gauge operator T can be linearized. To be precise, there are certain canonical choices of T in terms of eigenfunctions of the underlying scattering problem. The particular choice relevant for this paper is given by the following theorem.

Theorem 1: (a general Darboux theorem)

Let M be a differential operator, let χ be an invertible matrix solution of $\chi_t = P_0(M\chi)$. Then the operator

$$T = \chi\partial\chi^{-1} = \partial - \chi_x\chi^{-1} \quad , \quad T^{-1} = \chi\partial^{-1}\chi^{-1} \tag{3.7}$$

satisfies $T_t = -P_{<0}(TMT^{-1})T$ and the gauge transformed operator $\tilde{M} = TMT^{-1} + T_tT^{-1} = P_{\geq 0}(TMT^{-1})$ has no negative differential orders. Let Φ be a (matrix or vector) solution of $\Phi_t = P_0(M\Phi)$, then

$$\tilde{\Phi} = P_0(T\Phi) = \chi(\chi^{-1}\Phi)_x \tag{3.8}$$

solves the new linear equation $\tilde{\Phi}_t = P_0(\tilde{M}\tilde{\Phi})$. Let Ψ be a solution of the adjoint equation $\Psi_t = -P_0(M^*\Psi)$. Let $S(\Psi,\chi)$, $S(\Psi,\Phi)$ be the potentials of Lemma 1 satisfying

$$\begin{aligned} S(\Psi,\chi)_x &= \Psi^\dagger\chi \quad , \quad S(\Psi,\chi)_t = res(\partial^{-1}\Psi^\dagger M\chi\partial^{-1}) \quad , \\ S(\Psi,\Phi)_x &= \Psi^\dagger\Phi \quad , \quad S(\Psi,\Phi)_t = res(\partial^{-1}\Psi^\dagger M\Phi\partial^{-1}) \quad . \end{aligned} \tag{3.9}$$

Then

$$\tilde{\Psi}^\dagger = -S(\Psi,\chi)\chi^{-1} \quad , \tag{3.10}$$

is a solution of the new adjoint scattering problem $\tilde{\Psi}_t = -P_0(\tilde{M}^*\tilde{\Psi})$ and

$$\tilde{S}(\tilde{\Psi},\tilde{\Phi}) = S(\Psi,\Phi) - S(\Psi,\chi)\chi^{-1}\Phi = S(\Psi,\Phi) + \tilde{\Psi}^\dagger\Phi \tag{3.11}$$

is a potential for the new pair $\tilde{\Psi},\tilde{\Phi}$ satisfying

$$\tilde{S}(\tilde{\Psi},\tilde{\Phi})_x = \tilde{\Psi}^\dagger\tilde{\Phi} \quad , \quad \tilde{S}(\tilde{\Psi},\tilde{\Phi})_t = res(\partial^{-1}\tilde{\Psi}^\dagger\tilde{M}\tilde{\Phi}\partial^{-1}) \quad . \tag{3.12}$$

For the **proof** we observe

$$T_tT^{-1} = -\chi[\chi^{-1}\chi_t]_x\partial^{-1}\chi^{-1} \tag{3.13}$$

and

$$P_{<0}(TMT^{-1}) = P_{<0}(\chi\partial\chi^{-1}\ M\ \chi\partial^{-1}\chi^{-1}) = \chi P_0(\partial\chi^{-1}\ M\chi)\ \partial^{-1}\chi^{-1} = \chi[\chi^{-1}P_0(M\chi)]_x\partial^{-1}\chi^{-1} \quad . \tag{3.14}$$

Further, we have

$$\begin{aligned} \tilde{\Phi}_t &= P_0(T_t\Phi + T\Phi_t) = P_0(-P_{<0}(TMT^{-1})T\Phi + TP_0(M\Phi)) \\ &= P_0(-P_{<0}(TMT^{-1})T\Phi + TM\Phi) = P_0(P_{\geq 0}(TMT^{-1})T\Phi) = P_0(\tilde{M}\tilde{\Phi}) \end{aligned} \tag{3.15}$$

and

$$
\begin{aligned}
P_0(\tilde{M}^*\tilde{\Psi})^\dagger &= res(\partial^{-1}\tilde{\Psi}^\dagger\tilde{M}) = -res(\partial^{-1}S\chi^{-1}\chi\partial\chi^{-1}\ M\ \chi\partial^{-1}\chi^{-1}) \\
&= -res(\partial^{-1}(\partial S - S_x)\chi^{-1}\ M\ \chi\partial^{-1}\chi^{-1}) \\
&= -S\chi^{-1}res(M\chi\partial^{-1})\chi^{-1} + res(\partial^{-1}\Psi^\dagger M\chi\partial^{-1}\chi^{-1}) \\
&= -S\chi^{-1}P_0(M\chi)\chi^{-1} + res(\partial^{-1}\Psi^\dagger M\chi\partial^{-1})\chi^{-1} \\
&= -S\chi^{-1}\chi_t\chi^{-1} + S_t\chi^{-1} = (S\chi^{-1})_t = -\tilde{\Psi}_t^\dagger\ ,
\end{aligned}
\tag{3.16}
$$

where we have put $S = S(\Psi,\chi)$. Finally, for the potentials one has

$$
\tilde{S}(\tilde{\Psi},\tilde{\Phi})_x = S(\Psi,\Phi)_x - S(\Psi,\chi)_x\chi^{-1}\Phi - S(\Psi,\chi)(\chi^{-1}\Phi)_x = -S(\Psi,\chi)(\chi^{-1}\Phi)_x = \tilde{\Psi}^\dagger\tilde{\Phi} \tag{3.17}
$$

and

$$
\begin{aligned}
res(\partial^{-1}\tilde{\Psi}^\dagger\tilde{M}\tilde{\Phi}\partial^{-1}) &= -res(\partial^{-1}S\chi^{-1}\ \chi\partial\chi^{-1}\ M\ \chi\partial^{-1}\chi^{-1}\ \chi(\chi^{-1}\Phi)_x\partial^{-1}) \\
&= -res(\partial^{-1}\ (\partial S - S_x)\ \chi^{-1}\ M\ \chi\partial^{-1}(\partial\chi^{-1}\Phi - \chi^{-1}\Phi\partial)\partial^{-1}) \\
&= -S\chi^{-1}res(M\Phi\partial^{-1}) + S\chi^{-1}res(M\chi\partial^{-1})\chi^{-1}\Phi \\
&\qquad + res(\partial^{-1}\Psi^\dagger M\Phi\partial^{-1}) - res(\partial^{-1}\Psi^\dagger M\chi\partial^{-1})\chi^{-1}\Phi \\
&= -S(\Psi,\chi)\chi^{-1}\Phi_t + S(\Psi,\chi)\chi^{-1}\chi_t\chi^{-1}\Phi + S(\Psi,\Phi)_t - S(\Psi,\chi)_t\chi^{-1}\Phi \\
&= (\ S(\Psi,\Phi) - S(\Psi,\chi)\chi^{-1}\Phi\ )_t = \tilde{S}(\tilde{\Psi},\tilde{\Phi})_t\ .
\end{aligned}
\tag{3.18}
$$

This finishes the proof of Theorem 1.

This Darboux theorem states that an eigenfunction χ of the scattering problem associated with a differential operator M can be used to trigger a gauge transformation to a new $\tilde{M}$. The important feature is that further eigenfunctions and adjoint eigenfunctions for M can be transformed to new (adjoint) eigenfunctions for $\tilde{M}$ via the formulas (3.8) and (3.10). Hence, if several eigenfunctions $\chi_1, .., \chi_n$ for M are given, we may use χ_1 for a first gauge transformation from M to $\tilde{M}$. The transformation formula (3.8) applied to $\Phi = \chi_1$ yields a trivial new eigenfunction $\tilde{\chi}_1 = 0$, i.e. the eigenfunction used to trigger the transformation will be lost. However, the remaining $n-1$ eigenfunctions $\chi_2, .., \chi_n$ will be transformed to nontrivial new eigenfunctions $\tilde{\chi}_2, .., \tilde{\chi}_n$ for $\tilde{M}$. Thus, we can perform a second gauge transformation with $\tilde{\chi}_2$, say, which maps $\tilde{M}$ to a new operator with further eigenfunctions generated from $\tilde{\chi}_3, .., \tilde{\chi}_n$. The eigenfunction $\tilde{\chi}_2$ is mapped to 0 in this step. Iterating this process we may perform n transformations until all the eigenfunctions are "used up". With the following theorem we describe the form of the gauge transformed operator in a compact way with expressions involving only the original collection $\chi_1, .., \chi_n$ of eigenfunctions for the original operator M.

We start with a differential operator $M = M[0]$ and a (pseudo-differential) Lax operator $L = L[0]$ satisfying the Lax equation $L_t = [M, L]$. We assume that n eigenfunctions $\chi_i = \chi_i[0]$ solving $\chi_t = P_0(M\chi)$ are given. We also assume that a further eigenfunction $\Phi = \Phi[0]$ satisfying $\Phi_t = P_0(M\Phi)$ and an adjoint eigenfunction $\Psi = \Psi[0]$ satisfying $\Psi_t = -P_0(M^*\Psi)$ are given. We want to trace the transformations of Φ and Ψ in a repeated DT using the eigenfunctions $\chi_1, .., \chi_n$. In particular, for the transformation of the adjoint eigenfunction Ψ, we fix a set of potentials $S_i = S_i[0] = S(\Psi, \chi_i)$ defined as solutions of

$$S_{ix} = \Psi^\dagger \chi_i \quad , \quad S_{it} = res(\partial^{-1}\Psi^\dagger M \chi_i \partial^{-1}) \quad . \tag{3.19}$$

We fix the notation $T(\chi) = \chi\partial\chi^{-1}$ for the gauge operator and map the operator $M = M[0]$ to $M[1]=P_{\geq 0}(T(\chi_1[0])M[0]T(\chi_1[0])^{-1})$ in the first stage. This new operator is equipped with the eigenfunctions $\chi_2[1], .., \chi_n[1]$, $\Phi[1]$ and the adjoint eigenfunction $\Psi[1]$, which are the images of the corresponding objects under the transformation $T(\chi_1[0])$. We then use $\chi_2[1]$ to generate $M[2]$ with new (adjoint) eigenfunctions etc. In the step from level $[k-1]$ to $[k]$ the eigenfunction $\chi_k[k-1]$ will be used to trigger the transformation. Thus, the iteration of the Darboux Theorem 1 is given by the recursive definitions

$$\begin{aligned}
L[k] &= T(\chi_k[k-1])\ L[k-1]\ T(\chi_k[k-1])^{-1} \quad , \\
M[k] &= P_{\geq 0}(\ T(\chi_k[k-1])\ M[k-1]\ T(\chi_k[k-1])^{-1}\ ) \quad , \\
\chi_j[k] &= \chi_k[k-1]((\chi_k[k-1])^{-1}\chi_j[k-1])_x \quad , \quad j = k+1, .., n \quad , \\
\Phi[k] &= \chi_k[k-1]((\chi_k[k-1])^{-1}\Phi[k-1])_x \quad , \\
\Psi[k]^\dagger &= -S_k[k-1](\chi_k[k-1])^{-1} \quad , \\
S_j[k] &= S_j[k-1] + \Psi[k]^\dagger \chi_j[k-1] \quad , \quad j = 1, .., n \quad ,
\end{aligned} \tag{3.20}$$

where $S_j[k] = S(\Psi[k], \chi_j[k])$ is the potential for the pair $\Psi[k], \chi_j[k]$ associated with the operator $M[k]$. After n transformations we obtain the final Lax pair $\tilde{L} = L[n]$, $\tilde{M} = M[n]$, equipped with an eigenfunction $\tilde{\Phi} = \Phi[n]$ and an adjoint eigenfunction $\tilde{\Psi} = \Psi[n]$. The following theorem solves the recursion (3.20) and presents the final Lax pair $\tilde{L}, \tilde{M}$ and the (adjoint) eigenfunctions $\tilde{\Phi}$ and $\tilde{\Psi}$ in terms of the eigenfunctions $\chi_1, .., \chi_n$.

Theorem 2 (the iterated Darboux transformation)

Let $a_0, .., a_{n-1}$ and $b_1, .., b_n$ be the matrix functions determined as the solution of the linear algebraic system

$$\sum_{i=0}^{n} a_i \chi_j^{(i)} \ = \ 0 \quad , \quad j = 1, .., n \quad , \tag{3.21}$$

with $a_n = 1$ and

$$\sum_{j=1}^{n} \chi_j^{(i)} b_j \ = \ \begin{cases} 0 & \text{for } i = 0, .., n-2 \\ 1 & \text{for } i = n-1 \quad . \end{cases} \tag{3.22}$$

Then the Lax pair $\tilde{L}$, $\tilde{M}$ obtained by an iteration of the DT is given by the gauge transformation $\tilde{L} = T_n M T_n^{-1}$, $\tilde{M} = P_{\geq 0}(T_n M T_n^{-1})$ with

$$T_n = \sum_{i=0}^{n} a_i \partial^i \quad , \qquad T_n^{-1} = \sum_{j=1}^{n} \chi_j \partial^{-1} b_j \quad . \tag{3.23}$$

Eigenfunctions Φ and adjoint eigenfunctions Ψ for the original pair L, M are transformed into new (adjoint) eigenfunctions for the pair $\tilde{L}, \tilde{M}$ via

$$\tilde{\Phi} = \sum_{j=0}^{n} a_j \Phi^{(j)} \quad , \quad \tilde{\Psi}^{\dagger} = -\sum_{j=1}^{n} S(\Psi, \chi_j) b_j \quad . \tag{3.24}$$

The **proof** of the theorem is given in the appendix. We have, of course, implied that the matrix eigenfunction $\chi_k[k-1]$ encountered after the first $k-1$ transformations is invertible, so that it can be used to trigger the next transformations from level $[k-1]$ to $[k]$. The proof, in particular the decomposition (A.13), shows that this is a condition on the invertibility of the Wronskian matrix associated with the eigenfunctions $\chi_1, .., \chi_n$ used for the transformation. This condition is equivalent to the assumption that the equations (3.21) and (3.22) are solvable.

We remark that in the case of the scalar KP hierarchy, where all coefficients and (adjoint) eigenfunctions are scalar functions, the systems (3.21) and (3.22) can be solved via Cramer's rule and lead to compact expressions such as

$$a_i = \frac{(-1)^{(n-i)}}{\mathcal{W}(\chi_1, .., \chi_n)} \, det \begin{array}{c} \qquad\qquad\qquad \widehat{i} \\ \begin{pmatrix} \chi_1 & \chi_{1x} & .. & \chi_1^{(n)} \\ .. & .. & .. & .. \\ \chi_n & \chi_{nx} & .. & \chi_n^{(n)} \end{pmatrix} \end{array} \quad . \tag{3.25}$$

Here $\mathcal{W}(\chi_1, .., \chi_n)$ is the Wronskian determinant of the eigenfunctions and $\widehat{i}$ is to indicate that the column with the ith derivatives is omitted. In particular, the highest non-trivial coefficient a_{n-1} of the transformation operator $T_n = \partial^n + a_{n-1}\partial^{n-1} + .. + a_0$ is given by

$$a_{n-1} = -\frac{\mathcal{W}_x}{\mathcal{W}} = -\frac{d}{dx} \, ln \, \mathcal{W}(\chi_1, .., \chi_n) \quad . \tag{3.26}$$

In the scalar case also the solution of system (3.22) is given by a Wronskian formula

$$b_j = (-1)^{n-j} \frac{\mathcal{W}(\chi_1, .., \widehat{j}, .., \chi_n)}{\mathcal{W}(\chi_1, .., \chi_n)} \quad , \tag{3.27}$$

where $\widehat{j}$ indicates that the function χ_j is omitted in the argument list. Hence, for the scalar case, the transformation operator T_n and its inverse can be represented by formal determinants

$$T_n = \frac{1}{\mathcal{W}(\chi_1, .., \chi_n)} det \begin{pmatrix} \chi_1 & .. & \chi_n & 1 \\ \chi_{1x} & .. & \chi_{nx} & \partial \\ .. & .. & .. & .. \\ \chi_1^{(n)} & .. & \chi_n^{(n)} & \partial^n \end{pmatrix} \tag{3.28}$$

and

$$T_n^{-1} = (-1)^{n-1} det \begin{pmatrix} \chi_1 \partial^{-1} & \chi_1 & \chi_{1x} & .. & \chi_1^{(n-2)} \\ \chi_2 \partial^{-1} & \chi_2 & \chi_{2x} & .. & \chi_2^{(n-2)} \\ .. & .. & .. & .. & .. \\ \chi_n \partial^{-1} & \chi_n & \chi_{nx} & .. & \chi_n^{(n-2)} \end{pmatrix} \frac{1}{\mathcal{W}(\chi_1, .., \chi_n)} \quad . \tag{3.29}$$

For T_n the determinant is given as a formal expansion w.r.t. the last column (collecting all minors on the left of the differential symbols) and for T_n^{-1} an expansion w.r.t. the first column is understood (collecting all minors on the right of $\chi_j \partial^{-1}$). The transformation formulas (3.24) then obtain the simple form

$$\tilde{\Phi} = \frac{\mathcal{W}(\chi_1, .., \chi_n, \Phi)}{\mathcal{W}(\chi_1, .., \chi_n)} \ , \ \tilde{\Psi} = \frac{(-1)^n}{\mathcal{W}(\chi_1, .., \chi_n)} det \begin{pmatrix} S(\Psi, \chi_1) & \chi_1 & \chi_{1x} & .. & \chi_1^{(n-2)} \\ .. & .. & .. & .. & .. \\ S(\Psi, \chi_n) & \chi_n & \chi_{nx} & .. & \chi_n^{(n-2)} \end{pmatrix} \ . \tag{3.30}$$

4 The multicomponent KP hierarchy

The DT $L \to \tilde{L} = TLT^{-1}$, $M \to \tilde{M} = P_{\geq 0}(TMT^{-1})$ of the last section maps the Lax equation for the pair L, M to the same Lax equation for the new pair $\tilde{L}, \tilde{M}$. In specific realizations the operators L and M are parametrized by a set of fields $w = (w_1, w_2, ..)$, so that the Lax equation implies a nonlinear system for the fields w. In order to obtain an invariance of the nonlinear system we want to ensure that the gauge transformation on L and M leaves this parametrization invariant. To be precise, we want to find a parametrization $\mathcal{L}$ and $\mathcal{M}$ such that with $L = \mathcal{L}(w)$, $M = \mathcal{M}(w)$ the gauge transformed operators are given by the same parametrization $\tilde{L} = \mathcal{L}(\tilde{w})$, $\tilde{M} = \mathcal{M}(\tilde{w})$, but with transformed fields $\tilde{w} = (\tilde{w}_1, \tilde{w}_2, ..)$. A particular class of parametrizations for Lax pairs is given by Sato's construction of the (multicomponent) KP hierarchy. We will show that in this case the DT will indeed leave the parametrization of L and M invariant, as the Darboux theorem for the Lax pair can be traced back to a Darboux type transformation of the parametrizing fields w.

Following [23-27] we briefly review the construction of the multicomponent KP hierarchy. Let g_0 be the finite dimensional algebra, from which the coefficients of the following pseudo-differential operators are taken. The basic object to construct the integrable equations of the KP hierarchy is the following dressing operator

$$W = 1 + w_1 \partial^{-1} + w_2 \partial^{-2} + ... \quad , \quad w_j \in g_0 \ , \tag{4.1}$$

where the fields $w = (w_1, w_2, ..)$ will be the dynamical fields of the KP hierarchy satisfying a certain set of nonlinear differential equations. This operator can be inverted by using an ansatz $W^{-1} = 1 + v_1 \partial^{-1} + v_2 \partial^{-2} + ..$. The requirement $WW^{-1} = 1$ leads to a straightforward recursive scheme for the coefficients v_j, from which they can be expressed as differential expressions of the coefficients w_j in W. In particular, the first terms of the expansion of W^{-1} are given as

$$W^{-1} = 1 - w_1 \partial^{-1} - (w_2 - w_1^2) \partial^{-2} + ... \quad . \tag{4.2}$$

The coefficients w_j of W are to be endowed with a certain dynamics. For this we choose a collection of commuting elements in g_0, use them as coefficients of powers of the differential operator and define a set of commuting "undressed operators"

$$\mathcal{A} = \{ \ a_n \partial^n \ , \ n \in \mathbb{N} \ , \ a_n \in g_0 \ , \ [a_n, a_m] = 0 \ \} \ . \tag{4.3}$$

With each element $A \in \mathcal{A}$ we associate a dynamics of the dressing operator W by requiring

$$W_{t_A} = -P_{<0}(WAW^{-1})W = -WA + P_{\geq 0}(WAW^{-1})W \ , \tag{4.4}$$

where t_A is to denote the evolution parameter ("time") of this flow. We note that this does indeed define a consistent dynamics for all the coefficients w_j in W, as the right hand side of (4.4) is a pseudo-differential operator with highest differential order ∂^{-1}. The set of nonlinear equations for the coefficients w_j in W obtained from (4.4) is called the multicomponent KP hierarchy (associated with the collection $\mathcal{A}$ of undressed operators).

It will be convenient to rephrase the system (4.4) as a hierarchy of Lax equations. For this we fix an arbitrary element $C \in \mathcal{A}$ and define a Lax operator

$$L = WCW^{-1} \quad . \tag{4.5}$$

Further, one defines the differential operators

$$M_A = P_{\geq 0}(WAW^{-1}) \tag{4.6}$$

and observes

$$\begin{aligned} L_{t_A} &= -P_{<0}(WAW^{-1})WCW^{-1} + WCW^{-1}P_{<0}(WAW^{-1}) \\ &= [M_A - WAW^{-1}, L] = [M_A, L] - W[A, C]W^{-1} \end{aligned} \tag{4.7}$$

for any $A \in \mathcal{A}$. Hence, due to the commutativity of the undressed operators in $\mathcal{A}$, the equations (4.4) for the dressing operator W are transformed into the Lax form

$$L_{t_A} = [M_A, L] \quad . \tag{4.8}$$

In particular, for the choice $C = c\partial^N$ with some constant element $c \in g_0$ the Lax operator obtains the form

$$\begin{aligned} L = Wc\partial^N W^{-1} &= (1 + w_1\partial^{-1} + ..)c\partial^N(1 - w_1\partial^{-1} + ..) \\ &= c\partial^N + [w_1, c]\partial^{N-1} + u_{N-2}\partial^{N-2} + .. + u_0 + u_{-1}\partial^{-1} + .. \quad , \end{aligned} \tag{4.9}$$

where all coefficients u_j in (4.9) are given as differential expressions of the coefficients w_j of the dressing operator (4.1). This provides a convenient parametrization of the Lax operator in terms of the matrix functions w_j. Suitable M's for the evolution (4.8) are given by the operators M_A, which by construction are parametrized in terms of the functions w_j, too. This particular parametrization of L and M_A will guarantee, that the right hand side of (4.8) is automatically tangent to the operator (4.9).

The basic observation for the evolution equations (4.4)/(4.8) is the fact that the flows associated with two different undressed operators $A, B \in \mathcal{A}$ commute. In particular, observing the identities

$$(W_{t_A})_{t_B} - (W_{t_B})_{t_A} = -P_{<0}(W[A, B]W^{-1})W \quad , \quad M_{At_B} - M_{Bt_A} - [M_B, M_A] = P_{\geq 0}(W[A, B]W^{-1}) \quad , \tag{4.10}$$

one finds $(W_{t_A})_{t_B} = (W_{t_B})_{t_A}$ and $M_{At_B} - M_{Bt_A} = [M_B, M_A]$ for any choice for the commuting undressed operators $A, B \in \mathcal{A}$. As a consequence, also the derived Lax equations (4.8) commute for different choices of A and B in $\mathcal{A}$.

We introduce eigenfunctions Φ and adjoint eigenfunctions Ψ as solutions of the associated linear evolution equations

$$\Phi_{t_A} = P_0(M_A\Phi) \quad , \quad \Psi_{t_A} = -P_0(M_A^*\Psi) \quad . \tag{4.11}$$

Here we may either think of these functions as vectors (columns) or matrices. Due to the zero-curvature condition these linear flows may be considered simultaneously for the entire modified KP hierarchy, i.e. for all choices of $A \in \mathcal{A}$.

We now formulate the generalization of the classical Darboux theorem, which gives rise to an invariance of the KP hierarchy (4.4). For the scalar case it was shown in [7] that a Darboux theorem can be established for the whole KP hierarchy as a gauge transformation of the Lax equations (4.8). In [33] the corresponding formulation was given in terms of the evolution (4.4) of the dressing operator W, which is more convenient for the multicomponent case. We review the results of [33] with the following theorem:

Theorem 3: (the Darboux theorem for the KP hierarchy)

Let the dressing operator W satisfy the KP hierarchy (4.4), let χ be an invertible matrix eigenfunction satisfying $\chi_{t_A} = P_0(M_A\chi)$ with $M_A = P_{\geq 0}(WAW^{-1})$, $[\partial, A] = 0$. Then

$$\tilde{W} = \chi\partial\chi^{-1}W\partial^{-1} \tag{4.12}$$

satisfies again the KP hierarchy

$$\tilde{W}_{t_A} = -P_{<0}(\tilde{W}A\tilde{W}^{-1})\tilde{W} \quad . \tag{4.13}$$

The new dressing operator is again of the form $\tilde{W} = 1 + \sum_{j\geq 1} \tilde{w}_j\partial^{-j}$, where (with $w_0 = 1$) the new coefficients are given by

$$\tilde{w}_j = w_j + \chi(\chi^{-1}w_{j-1})_x \quad , \quad j \in \mathbb{N} \ . \tag{4.14}$$

Proof: We first consider $\hat{W} = TW$, where $T = \chi\partial\chi^{-1}$ is the gauge operator of the DT satisfying $T_{t_A} = -P_{<0}(TM_AT^{-1})T$ (Theorem 1). The dynamics of $\hat{W}$ is given by

$$\begin{aligned} \hat{W}_{t_A} &= -P_{<0}(TM_AT^{-1})TW - TP_{<0}(WAW^{-1})W \\ &= -TM_AW + P_{\geq 0}(TM_AT^{-1})TW - T(WAW^{-1})W + TP_{\geq 0}(WAW^{-1})W \\ &= P_{\geq 0}(TP_{\geq 0}(WAW^{-1})T^{-1})\hat{W} - \hat{W}A \quad . \end{aligned} \tag{4.15}$$

We observe that the negative orders of WAW^{-1} will not contribute to $P_{\geq 0}(T\ WAW^{-1}\ T^{-1})$, so that we may write

$$\hat{W}_{t_A} = P_{\geq 0}(TWAW^{-1}T^{-1})\hat{W} - \hat{W}A = (P_{\geq 0}(\hat{W}A\hat{W}^{-1}) - \hat{W}A\hat{W}^{-1})\hat{W} = -P_{<0}(\hat{W}A\hat{W}^{-1})\hat{W} \ . \tag{4.16}$$

Hence, $\hat{W}$ again satisfies the KP hierarchy (4.4). However, $\hat{W} = TW = (\chi\partial\chi^{-1})(1 + w_1\partial^{-1} + ..) = \partial + w_1 - \chi_x\chi^{-1} + (..)\partial^{-1} + ..$ has the leading order ∂ and is not of the form (4.1) any more. We observe that multiplication from the right with any constant operator commuting with A leaves (4.4) invariant. Hence, $\tilde{W} = \hat{W}\partial^{-1}$ also satisfies the KP hierarchy (4.4). The leading order of this dressing operator $\tilde{W} = 1 + (w_1 - \chi_x\chi^{-1})\partial^{-1} + ..$ is 1 and all coefficients $\tilde{w}_j$ of $\tilde{W}$ are obtained from

$$\tilde{W} = \chi\partial\chi^{-1} \sum_{j\geq 0} w_j\partial^{-j-1} = \sum_{j\geq 0} w_j\partial^{-j} + \sum_{j\geq 0} \chi(\chi^{-1}w_j)_x\partial^{-j-1} \quad . \tag{4.17}$$

This finishes the proof of Theorem 3.

We had constructed Lax pairs $L = WCW^{-1}$, $M_A = P_{\geq 0}(WAW^{-1})$ from the dressing operator W, where L and M_A are parametrized in terms of the coefficients $w = (w_1, w_2, ..)$ in W. If we construct the corresponding operators $\tilde{L} = \tilde{W}C\tilde{W}^{-1}$, $\tilde{M}_A = P_{\geq 0}(\tilde{W}A\tilde{W}^{-1})$ from the transformed dressing operator $\tilde{W}$, these operators are parametrized in exactly the same way by the fields $\tilde{w} = (\tilde{w}_1, \tilde{w}_2, ..)$. We observe that the new Lax pair $\tilde{L}, \tilde{M}$ is given by

$$\begin{aligned} \tilde{L} &= \tilde{W}C\tilde{W}^{-1} = TWCW^{-1}T^{-1} = TLT^{-1} \quad , \\ \tilde{M}_A &= P_{\geq 0}(\tilde{W}A\tilde{W}^{-1}) = P_{\geq 0}(TWAW^{-1}T^{-1}) = P_{\geq 0}(TP_{\geq 0}(WAW^{-1})T^{-1}) = P_{\geq 0}(TM_AT^{-1}) \quad , \end{aligned} \tag{4.18}$$

so that the transformation (4.12) implies the gauge transformation

$$L \ \rightarrow \ \tilde{L} \ = \ TLT^{-1} \quad , \quad M_A \ \rightarrow \ \tilde{M}_A \ = \ P_{\geq 0}(TM_AT^{-1}) \tag{4.19}$$

considered in Section 3. Hence, the general Darboux Theorems 1 and 2 apply to the Lax pairs encountered in the KP hierarchy. The additional feature for the KP hierarchy is the fact that the DT leaves the form of these operators invariant, if we parametrize them in terms of the coefficients of the dressing operator.

We finally note that the iteration of the DT given by Theorem 2 leads to the transformation

$$W \ \rightarrow \ \tilde{W} = T_nW\partial^{-n} \tag{4.20}$$

of the KP hierarchy, where $T_n = \sum_{i=0}^{n-1} a_i\partial^i + \partial^n$ is given by the solutions of (3.21) with eigenfunctions $\chi_1, .., \chi_n$ satisfying the linear equations $\chi_{t_A} = P_0(M_A\chi)$. The new coefficients $\tilde{w}_1, \tilde{w}_2, ..$ can be expressed in terms of the old coefficients $w_1, w_2, ..$ and the "Wronskian expressions" a_i by collecting the differential orders in

$$\tilde{W} = \sum_{j=0}^{\infty} \tilde{w}_j\partial^{-j} = \sum_{i=0}^{n}\sum_{j=0}^{\infty} a_i\partial^i \ w_j \ \partial^{-j-n} = \sum_{i=0}^{n}\sum_{j=0}^{\infty}\sum_{k=0}^{i} \begin{pmatrix} i \\ k \end{pmatrix} \ a_i \ w_j^{(i-k)} \ \partial^{k-j-n} \quad . \tag{4.21}$$

One has $\tilde{w}_0 = w_0 = 1$ and for $j \geq 1$ various re-summations lead to

$$\tilde{w}_j = \sum_{i=0}^{min(n,j)} \ \sum_{k=i}^{min(n,j)} \begin{pmatrix} n+i-k \\ i \end{pmatrix} a_{n+i-k} \ w_{j-k}^{(i)} \tag{4.22}$$

$$= w_j \; + \sum_{i=1}^{min(n,j)} \begin{pmatrix} n \\ i \end{pmatrix} w^{(i)}_{j-i} \; + \sum_{i=0}^{min(n,j)-1} \; \sum_{k=i+1}^{min(n,j)} \begin{pmatrix} n+i-k \\ i \end{pmatrix} a_{n+i-k} \, w^{(i)}_{j-k}$$

as the new solution of the KP hierarchy. With (4.22) this new solution is expressed in terms of the seed solution w_j and the Wronskian expressions a_i of the eigenfunctions $\chi_1, .., \chi_n$.

If we start with the trivial solution $W = 1$ (i.e. $w_1 = w_2 = .. = 0$), then the Darboux transformed solution is simply given by

$$\tilde{W} = T\partial^{-n} = (\partial^n + a_{n-1}\partial^{n-1} + .. + a_0)\partial^{-n} = 1 + a_{n-1}\partial^{-1} + .. + a_0\partial^{-n} \quad , \tag{4.23}$$

where $a_0, .., a_{n-1}$ are the Wronskian expressions of a collection of eigenfunctions $\chi_1, .., \chi_n$ satisfying the simple constant coefficient equations $\chi_{t_A} = P_0(A\chi)$.

5 Constraints of the KP hierarchy

We want to apply the results of the Darboux Theorems 1-3 to certain constraints of the KP hierarchy. For the case of the scalar KP hierarchy constraints in terms of eigenfunctions and adjoint eigenfunctions were considered in a series of papers ([16-22,29-32]). In this section we will exhibit a systematic approach to these reductions, which works for the multicomponent case and reduces to the established results for the scalar case.

We first establish the general result that pairs of eigenfunctions and adjoint eigenfunctions can be used to characterize a one-parameter symmetry transformation of (4.4) and (4.8) generated by "squared eigenfunctions". If z is the parameter, the generator of this symmetry is given by the flow W_z given in the following theorem.

Theorem 4: (squared eigenfunctions as symmetries of the KP hierarchy)

Let Φ_i, Ψ_i, $i = 1, .., m$, be eigenfunctions and adjoint eigenfunctions of the KP hierarchy (4.4) satisfying (4.11). Then the evolution of W given by

$$W_z \;=\; - \sum_{i=1}^{m} \Phi_i \partial^{-1} \Psi_i^{\dagger} \, W \tag{5.1}$$

is compatible with the KP hierarchy (4.4).

For the **proof** we use the definitions of W_{t_A} and W_z and obtain

$$\begin{aligned}(W_z)_{t_A} - (W_{t_A})_z \;=\; (\; & - \textstyle\sum_i P_0(M_A\Phi_i)\partial^{-1}\Psi_i^{\dagger} \; + \sum_i \Phi_i\partial^{-1}P_0(M_A^*\Psi_i)^{\dagger} \\ & + [\; \textstyle\sum_i \Phi_i\partial^{-1}\Psi_i^{\dagger} \, , P_{<0}(WAW^{-1})] - P_{<0}([\; \sum_i \Phi_i\partial^{-1}\Psi_i^{\dagger} \, , WAW^{-1}] \;) \;) \; W \quad .\end{aligned} \tag{5.2}$$

Making use of the identities (2.10) one finds

$$
\begin{aligned}
&P_{<0}([\ \textstyle\sum_i \Phi_i \partial^{-1} \Psi_i^\dagger\ , WAW^{-1}]) \\
&= \textstyle\sum_i \Phi_i \partial^{-1} P_0(M_A^* \Psi_i)^\dagger - \sum_i P_0(M_A \Phi_i) \partial^{-1} \Psi_i^\dagger + [\ \sum_i \Phi_i \partial^{-1} \Psi_i^\dagger\ , P_{<0}(WAW^{-1})]\ \ ,
\end{aligned}
\tag{5.3}
$$

so that $(W_z)_{t_A} - (W_{t_A})_z = 0$ and Theorem 4 is proven.

We remark that in terms of Lax equations the hierarchy (4.8) has the additional symmetry

$$
L_z = -[\ \sum_{i=1}^{m} \Phi_i \partial^{-1} \Psi_i^\dagger\ , L] \tag{5.4}
$$

which is a direct consequence of the definition $L = WCW^{-1}$ and the evolution (5.1). This flow commutes with all evolution equations (4.8).

We briefly show that Theorem 4 provides established results for the scalar KP hierarchy as a special case. For the scalar KP hierarchy it was observed ([14, 15]) that squared eigenfunctions (i.e. products of eigenfunctions and adjoint eigenfunctions) are conserved covariants, so that $\sum_i (\Phi\Psi)_x$ generates a symmetry of the scalar KP hierarchy. In the scalar case it is convenient to choose $\mathcal{A} = \{\partial^n\ ,\ n \in \mathbb{N}\}$ and $L = W\partial W^{-1} = \partial + u_1 \partial^{-1} + u_2 \partial^{-2} + \ldots\ ,$ so that the KP hierarchy (4.8) can be rephrased as $L_{t_n} = [P_{\geq 0}(W\partial^n W^{-1}), L] = [P_{\geq 0}(L^n), L]$ (see e.g. [27] for a simple introduction). In particular, the field u_1 satisfies the scalar KP equation and its higher flows. In this case (5.4) implies

$$
L_z = -[\ \sum_i \Phi_i \partial^{-1} \Psi_i^\dagger\ , \partial + u_1 \partial^{-1} + \ldots] = \sum_i (\Phi_i \Psi_i^\dagger)_x\ \partial^{-1} +\ \textit{lower orders}\ \ , \tag{5.5}
$$

where the transposition of the adjoint eigenfunctions is irrelevant in the scalar case. Hence, the symmetry (5.4) implies $u_{1z} = \sum_i (\Phi_i \Psi_i)_x$ for the solution u_1 of the scalar KP equation, i.e. the x-derivative of the squared eigenfunction $\sum_i \Phi_i \Psi_i$ generates a symmetry of the KP equation and its higher flows. It seems that Theorem 4 and the corresponding Lax representation (5.4) gives the general formulation of this result which applies to the multicomponent case, too.

We will now consider constrained Lax operators of the form (1.1). We use the construction of Lax operators from the KP hierarchy given by (4.9), and we will show that the negative part of (4.9) can be constrained to the form $\sum_i \Phi_i \partial^{-1} \Psi_i^\dagger$ with eigenfunctions Φ_i and adjoint eigenfunctions Ψ_i.

Lemma 4: (constrained KP flows)

For any $A, C \in \mathcal{A}$ the constraint

$$
P_{<0}(WCW^{-1}) = \sum_{i=1}^{m} \Phi_i \partial^{-1} \Psi_i^\dagger \tag{5.6}
$$

is left invariant by the dynamics

$$
W_{t_A} = -P_{<0}(WAW^{-1})W\ \ ,\ \ \Phi_{it_A} = P_0(M_A \Phi_i)\ \ ,\ \ \Psi_{it_A} = -P_0(M_A^* \Psi_i)\ \ ,\ \ i = 1, .., m\ . \tag{5.7}
$$

As a consequence, with the choice $C = c\partial^N$, the Lax operator $L = WCW^{-1}$ has the form

$$L = c\partial^N + [w_1, c]\partial^{N-1} + u_{N-2}\partial^{N-2} + .. + u_0 + \sum_{i=1}^{m} \Phi_i \partial^{-1} \Psi_i^\dagger \quad , \tag{5.8}$$

which is preserved under the flows $L_{t_A} = [M_A, L]$.

For the **proof** we only have to verify

$$\frac{d}{dt_A} \left(P_{<0}(WCW^{-1}) - \sum_i \Phi_i \partial^{-1} \Psi_i^\dagger \right) = 0 \quad , \tag{5.9}$$

when W, Φ_i, Ψ_i evolve according to (5.7). This is readily checked by a direct computation. When we introduce the Lax operator $L = WCW^{-1}$, its pseudo-differential part is by construction given by (5.6). For $C = c\partial^N$ all coefficients of the differential part $P_{\geq 0}(L) = \sum_{i=0}^N u_i \partial^i$ can be calculated in terms of the coefficients of the dressing operator, and (4.9) takes care of the form of the highest differential orders. We remark that the constraint (5.6) leads to

$$W_{t_C} \;=\; -\, P_{<0}(WCW^{-1})W \;=\; -\sum_{i=1}^{m} \Phi_i \partial^{-1} \Psi_i^\dagger \;\, W \;=\; W_z \tag{5.10}$$

and

$$\begin{aligned} L_{t_C} = [M_C, L] \;&=\; [P_{\geq 0}(WCW^{-1}), WCW^{-1}] \\ &=\; -[P_{<0}(WCW^{-1}), WCW^{-1}] \;=\; -[\; \textstyle\sum_i \Phi_i \partial^{-1} \Psi_i^\dagger \;, L] = L_z \quad , \end{aligned} \tag{5.11}$$

so that the reduction of the KP hierarchy (4.4) via the constraint (5.6) can be regarded as a symmetry reduction

$$W_{t_C} = W_z \quad , \quad L_{t_C} = L_z \tag{5.12}$$

involving the symmetry (5.1)/(5.4) generated by squared eigenfunctions. This provides a simple argument for the validity of Lemma 4.

6 The Darboux transformation for the constrained KP hierarchy

We now formulate the DT for the KP hierarchy subject to the symmetry constraint (5.6) of the last Section. This is most conveniently done in terms of Lax operators of the form (5.8). We drop the index A labeling the individual members of the KP hierarchy and consider the flow $L_t = [M, L]$ generated by some differential operator M. The general result of Theorem 1 guarantees that for the gauge transformation $L \to TLT^{-1}$, $M \to P_{\geq 0}(TMT^{-1})$ the Lax equation as well as the dynamics of eigenfunctions and adjoint eigenfunctions will be preserved. The following theorem concerns the "spatial" part of the DT in that we have to ensure that the form (5.8) of the constrained Lax operator is not changed under the DT. Up to now we have triggered the gauge transformation with any eigenfunction χ characterized only by its time evolution $\chi_t = P_0(M\chi)$. According to Theorem 1 this is sufficient to take care of all "temporal" aspects of the transformations. In order to preserve the form (5.8) an additional "spatial" condition has to be imposed on the triggering eigenfunction χ. It turns out that this condition is given by the spectral problem of Lemma 3.

Theorem 5: (the Darboux transformation for the constrained KP hierarchy)

Let L be of the form

$$L = u_N \partial^N + u_{N-1}\partial^{N-1} + .. + u_0 + \sum_{i=1}^{m} \Phi_i \partial^{-1} \Psi_i^\dagger \quad , \tag{6.1}$$

where $u_N, .., u_0$ are matrix functions and Φ_i, Ψ_i are either matrix or vector functions of space and time. Let χ be a matrix solution of the scattering problem

$$P_0(L\chi) + \sum_{i=1}^{m} \Phi_i S(\Psi_i, \chi) = \chi\, \Lambda \quad , \tag{6.2}$$

where Λ is a constant spectral matrix and $S(\Psi_i, \chi)$ are functions satisfying $S(\Psi_i, \chi)_x = \Psi_i^\dagger \chi$.

a) The gauge transformed operator $\tilde{L} = TLT^{-1}$ with $T = \chi\partial\chi^{-1}$ has again the form

$$\tilde{L} = \tilde{u}_N \partial^N + \tilde{u}_{N-1}\partial^{N-1} + .. + \tilde{u}_0 + \sum_{i=1}^{m} \tilde{\Phi}_i \partial^{-1} \tilde{\Psi}_i^\dagger \quad , \tag{6.3}$$

with

$$\tilde{\Phi}_i = \chi(\chi^{-1}\Phi_i)_x \quad , \quad \tilde{\Psi}_i^\dagger = -S(\Psi_i, \chi)\chi^{-1} \quad . \tag{6.4}$$

The coefficients $\tilde{u}_i$ with $i = 0, .., N$ can be expressed in the form

$$\tilde{u}_N = u_N \quad , \quad \tilde{u}_i = u_i + \mathcal{F}_i(u_{i+1}, .., u_N, \chi) \quad , \tag{6.5}$$

with suitable differential expressions $\mathcal{F}_i$ of the indicated arguments (see Theorem 6).

b) Let Φ be a solution of the spectral problem

$$P_0(L\Phi) + \sum_{i=1}^{m} \Phi_i S(\Psi_i, \Phi) = \Phi\, \Xi \tag{6.6}$$

with some constant spectral matrix Ξ and functions $S(\Psi_i, \Phi)$ satisfying $S(\Psi_i, \Phi)_x = \Psi_i^\dagger \Phi$. Then $\tilde{\Phi} = \chi(\chi^{-1}\Phi)_x$ satisfies the spectral problem

$$P_0(\tilde{L}\tilde{\Phi}) + \sum_{i=1}^{m} \tilde{\Phi}_i \tilde{S}(\tilde{\Psi}_i, \tilde{\Phi}) = \tilde{\Phi}\, \Xi \quad , \tag{6.7}$$

with $\tilde{S}(\tilde{\Psi}_i, \tilde{\Phi}) = S(\Psi_i, \Phi) + \tilde{\Psi}_i^\dagger \Phi$. These functions satisfy $\tilde{S}(\tilde{\Psi}_i, \tilde{\Phi})_x = \tilde{\Psi}_i^\dagger \tilde{\Phi}$.

For the **proof** of part a) of the theorem we first note that the positive differential order terms of $\tilde{L}$ are determined only be the positive differential order terms of L, i.e. $P_{\geq 0}(\tilde{L}) = P_{\geq 0}(TP_{\geq 0}(L)T^{-1})$. In particular, the highest differential order term of $\tilde{L}$ is generated by the highest differential order term $u_N\partial^N$, which leads to

$$\chi\partial\chi^{-1}\ u_N\partial^N\ \chi\partial^{-1}\chi^{-1} = u_N\partial^N + \ \mathit{lower\ orders} \quad . \tag{6.8}$$

Hence $\tilde{L}$ has the same highest differential order term as L. For the form (6.5) of the relations between the other coefficients we refer to the following theorem, where such relations are computed in the more general case of the iterated DT. It remains to show that the negative differential orders of $\tilde{L}$ are given by $\sum_i \tilde{\Phi}_i\partial^{-1}\tilde{\Psi}_i^\dagger$. With $S_i = S(\Psi_i, \chi)$ one calculates

$$\begin{aligned}
P_{<0}(\tilde{L}) &= P_{<0}(\chi\partial\chi^{-1}\ P_{\geq 0}(L)\ \chi\partial^{-1}\chi^{-1}) + \textstyle\sum_i\ \chi\partial\chi^{-1}\ \Phi_i\partial^{-1}\Psi_i^\dagger\ \chi\partial^{-1}\chi^{-1} \\
&= \chi P_0(\partial\chi^{-1}P_{\geq 0}(L)\chi)\partial^{-1}\chi^{-1} + \textstyle\sum_i\ \tilde{\Phi}_i\partial^{-1}\Psi_i^\dagger\ \chi\partial^{-1}\chi^{-1} + \textstyle\sum_i\ \Phi_i\Psi_i^\dagger\ \chi\partial^{-1}\chi^{-1} \\
&= \chi[\chi^{-1}P_0(L\chi)]_x\partial^{-1}\chi^{-1} + \textstyle\sum_i\ \tilde{\Phi}_i\partial^{-1}(\partial S_i - S_i\partial)\partial^{-1}\chi^{-1} + \textstyle\sum_i\ \Phi_i S_{ix}\partial^{-1}\chi^{-1} \\
&= \chi[\chi^{-1}P_0(L\chi)]_x\partial^{-1}\chi^{-1} + \textstyle\sum_i\ (\ \tilde{\Phi}_i S_i\partial^{-1}\chi^{-1} - \tilde{\Phi}_i\partial^{-1}S_i\chi^{-1} + \Phi_i S_{ix}\partial^{-1}\chi^{-1}\ ) \\
&= \textstyle\sum_i\ \tilde{\Phi}_i\partial^{-1}\tilde{\Psi}_i^\dagger + \chi[\chi^{-1}(P_0(L\chi) + \sum_i \Phi_i S_i)]_x\partial^{-1}\chi^{-1} \\
&= \textstyle\sum_i\ \tilde{\Phi}_i\partial^{-1}\tilde{\Psi}_i^\dagger + \chi\Lambda_x\partial^{-1}\chi^{-1} = \sum_i\ \tilde{\Phi}_i\partial^{-1}\tilde{\Psi}_i^\dagger \quad .
\end{aligned} \tag{6.9}$$

For part b) of the theorem we observe that only the differential part $l = P_{\geq 0}(L)$ contributes to $P_0(\tilde{L}\tilde{\Phi}) = P_0(TlT^{-1}\tilde{\Phi})$, so that

$$\begin{aligned}
P_0(\tilde{L}\tilde{\Phi}) &= P_0(\ \chi\partial\chi^{-1}\ l\ \chi\partial^{-1}\chi^{-1}\ \chi(\chi^{-1}\Phi)_x\ ) \\
&= \chi P_0(\ \partial\chi^{-1}\ l\ \chi\partial^{-1}(\partial\chi^{-1}\Phi - \chi^{-1}\Phi\partial)\ ) \\
&= \chi P_0(\partial\chi^{-1}l\Phi) - \chi P_0(\ \partial\chi^{-1}\ l\ \chi\partial^{-1}\chi^{-1}\Phi\partial\ ) \\
&= \chi P_0(\partial\chi^{-1}l\Phi) - \chi P_0(\partial\chi^{-1}l\chi)\ \chi^{-1}\Phi \\
&= \chi[\chi^{-1}P_0(l\Phi)]_x - \chi[\chi^{-1}P_0(l\chi)]_x\ \chi^{-1}\Phi \\
&= \chi[\chi^{-1}\Phi\Xi - \textstyle\sum_i \chi^{-1}\Phi_i S(\Psi_i,\Phi)]_x - \chi[\Lambda - \sum_i \chi^{-1}\Phi_i S(\Psi_i,\chi)]_x\ \chi^{-1}\Phi \\
&= \tilde{\Phi}\Xi - \textstyle\sum_i \chi[\chi^{-1}\Phi_i]_x\ (S(\Psi_i,\Phi) - S(\Psi_i,\chi)\chi^{-1}\Phi)\ - \sum_i \Phi_i S(\Psi_i,\Phi)_x + \sum_i \Phi_i S(\Psi_i,\chi)_x\chi^{-1}\Phi \\
&= \tilde{\Phi}\Xi - \textstyle\sum_i \tilde{\Phi}_i\tilde{S}(\tilde{\Psi}_i,\tilde{\Phi}) \quad .
\end{aligned} \tag{6.10}$$

This finishes the proof of Theorem 5.

We note that according to Lemma 3 we may combine the temporal aspects (Theorem 1) and the spatial aspects (Theorem 5) of the Darboux theorem. For a Lax equation $L_t = [M, L]$ we consider an eigenfunction χ which satisfies both the corresponding time evolution as well as the spectral problem

$$\chi_t = P_0(M\chi) \quad , \quad P_0(L\chi) + \sum_{i=1}^{m} \Phi_i S(\Psi_i,\chi) = \chi\ \Lambda \quad . \tag{6.11}$$

The DT triggered by χ will map any eigenfunction Φ satisfying both equations

$$\Phi_t = P_0(M\Phi) \quad , \quad P_0(L\Phi) + \sum_{i=1}^{m} \Phi_i S(\Psi_i, \Phi) = \Phi \, \Xi \tag{6.12}$$

to a new eigenfunction $\tilde{\Phi}$ for the transformed Lax pair $\tilde{L}, \tilde{M}$, which satisfies both the new evolution equation $\tilde{\Phi}_t = P_0(\tilde{M}\tilde{\Phi})$ as well as the new spectral problem $P_0(\tilde{L}\tilde{\Phi}) + \sum_{i=1}^{m} \tilde{\Phi}_i \tilde{S}(\tilde{\Psi}_i, \tilde{\Phi}) = \tilde{\Phi} \, \Xi$. The new potentials $\tilde{S}(\tilde{\Psi}_i, \tilde{\Phi})$ in Theorem 5 coincide with the new potentials $\tilde{S}$ in Theorem 1.

Hence, we may again consider the iteration of the DT given by Theorem 2, where we have started with a collection of eigenfunctions $\chi_1, .., \chi_n$ satisfying the time evolution $\chi_t = P_0(M\chi)$. We now assume that the eigenfunctions also satisfy the spectral problems

$$P_0(L\chi_j) + \sum_{i=1}^{m} \Phi_i S(\Psi_i, \chi_j) = \chi_j \, \Lambda_j \tag{6.13}$$

with constant spectral matrices Λ_j. The iteration of the Darboux theorem was given by the recursive scheme (3.20). In the step from level $[k-1]$ to $[k]$ triggered by the eigenfunction $\chi_k[k-1]$ we may use Theorem 5 (with $L[k-1]$ and $\Phi = \chi_j[k-1]$) to show that $\chi_j[k]$ again satisfies the spectral problem (6.2) with $L[k]$ and the eigenvalue matrix Λ_j. Hence, by induction, at all levels the Lax operators $L[k]$ are of the form

$$L[k] = u_N[k]\partial^N + u_{N-1}[k]\partial^{N-1} + .. + u_0[k] + \sum_{i=1}^{m} \Phi_i[k]\partial^{-1}(\Psi_i[k])^\dagger \quad , \tag{6.14}$$

and finally lead to a Lax operator $\tilde{L} = L[n]$ of the form (6.3) with $\tilde{u}_i = u_i[n]$, $\tilde{\Phi}_i = \Phi_i[n]$ and $\tilde{\Psi}_i = \Psi_i[n]$.

It remains to work out the formulas for $\tilde{u}_i$, $\tilde{\Phi}_i$, $\tilde{\Psi}_i$ in terms of the eigenfunctions $\chi_1, .., \chi_n$ used to trigger the transformations. For $\tilde{\Phi}_i$ this formula is given by (3.24.i) of Theorem 2 (with $\Phi = \Phi_i$), as the iterated transformation of an eigenfunction is only based on the recursion (3.20.iv), which on each level coincides with the transformation $\Phi_i \to \tilde{\Phi}_i$ in Theorem 5. Also for $\tilde{\Psi}_i$ the formula given by (3.24.ii) of Theorem 2 (with $\Psi = \Psi_i$) yields the correct result, as this formula is based on the recursion (3.20.v) and (3.20.vi). This, on each level, coincides with the transformation $\Psi_i \to \tilde{\Psi}_i$ in Theorem 5. Hence, it is only left to derive compact formulas for the coefficients $\tilde{u}_N, .., \tilde{u}_0$ in the differential part of the gauge transformed operator. We summarize these results with the following theorem.

Theorem 6 (the iterated Darboux transformation for the constrained KP hierarchy)

Let L be of the form (6.1), let $\chi_1, .., \chi_n$ satisfy (6.13) with constant spectral matrices Λ_j and functions $S(\Psi_i, \chi_j)$ satisfying $S(\Psi_i, \chi_j)_x = \Psi_i^\dagger \chi_j$, $i = 1, .., m$, $j = 1, .., n$. As in Theorem 2, let $a_0, .., a_{n-1}$ and $b_1, .., b_n$ be the matrix functions determined as the solution of the linear algebraic equations $\sum_{i=0}^{n} a_i \chi_j^{(i)} = 0$ with $a_n = 1$ and $\sum_{j=1}^{n} \chi_j^{(i)} b_j = \delta_{i,n-1}$. Then the operator $\tilde{L} = T_n L T_n^{-1}$ with T_n given by (3.23) is of the form (6.3). Here,

$$\tilde{\Phi}_i = \sum_{j=0}^{n} a_j \Phi_i^{(j)} \quad , \quad \tilde{\Psi}_i^\dagger = -\sum_{j=1}^{n} S(\Psi_i, \chi_j) b_j \quad , \quad i = 1, .., m \quad , \tag{6.15}$$

and the coefficients $\tilde{u}_N, .., \tilde{u}_0$ are given by the recursion

$$
\begin{aligned}
\tilde{u}_N &= u_N \ , \\
\tilde{u}_{N-j} &= \sum_{l=0}^{j} \sum_{k=K(l,n)}^{l} \begin{pmatrix} n+k-l \\ j-l \end{pmatrix} a_{n+k-l} u_{N-k}^{(j-l)} \\
&\quad - \sum_{l=1}^{j} \sum_{k=K(l,n)}^{l-1} \begin{pmatrix} N-k \\ j-l \end{pmatrix} \tilde{u}_{N-k} a_{n+k-l}^{(j-l)} \ , \quad j = 1, .., N \ ,
\end{aligned}
\tag{6.16}
$$

with $K(l,n) = max(0, l-n)$.

Proof: It has already been argued the the transformation formulas (3.24) of Theorem 2 can be applied to Φ_i and Ψ_i to obtain (6.15). We only have to deduce the recursion formulas (6.16) for the coefficients $\tilde{u}_N, .., \tilde{u}_0$ parametrizing the differential part of the Lax operator. We might use the form (3.23) of T_n and T_n^{-1} to calculate the various differential orders of $\tilde{L} = T_n L T_n^{-1}$. This would lead to a representation of the coefficients $\tilde{u}_j$ in $\tilde{L}$ in terms of the original coefficients u_j and the matrices $a_0, .., a_{n-1}$ as well as $b_1, .., b_n$. Both the a_i's and the b_j's encode the eigenfunctions $\chi_1, .., \chi_n$ used for the DT. For convenience, we choose to represent the $\tilde{u}_j$'s in terms of the a_i's only. We observe that we may rewrite the definition of the gauge transformation leading to $\tilde{L}$ as the condition $\tilde{L}T_n = T_n L$, so that only the a_i's parametrizing T_n will enter this equation. In order to obtain the $N+1$ coefficients $\tilde{u}_0, .., \tilde{u}_N$ we will need $N+1$ equations, which are provided by the differential orders $n, .., n+N$ of the relation $\tilde{L}T_n = T_n L$. Inserting $\tilde{L} = \sum_k \tilde{u}_k \partial^k +$ *negative orders* and $T_n = \sum_i a_i \partial^i$ and using various re-summations one obtains

$$
\begin{aligned}
P_{\geq n}(\tilde{L}T_n) &= P_{\geq n}(\sum_{k=0}^{N} \sum_{i=0}^{n} \tilde{u}_k \partial^k a_i \partial^i \ ) = P_{\geq n}(\sum_{k=0}^{N} \sum_{i=0}^{n} \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} \tilde{u}_k a_i^{(j)} \partial^{i+k-j} \ ) \\
&= \sum_{j=0}^{N} \ (\sum_{l=0}^{j} \sum_{k=K(l,n)}^{l} \begin{pmatrix} N-k \\ j-l \end{pmatrix} \tilde{u}_{N-k} a_{n+k-l}^{(j-l)} \ ) \ \partial^{N+n-j}
\end{aligned}
\tag{6.17}
$$

with $K(l,n) = max(0, l-n)$. We isolate the terms given by $k = l$ and use $a_n = 1$ to obtain

$$
P_{\geq n}(\tilde{L}T_n) = \sum_{j=0}^{N} \tilde{u}_{N-j} \partial^{N+n-j} \ + \ \sum_{j=1}^{N} \ (\sum_{l=1}^{j} \sum_{k=K(l,n)}^{l-1} \begin{pmatrix} N-k \\ j-l \end{pmatrix} \tilde{u}_{N-k} a_{n+k-l}^{(j-l)} \ ) \ \partial^{N+n-j} \ . \tag{6.18}
$$

On the other hand one finds

$$
\begin{aligned}
P_{\geq n}(T_n L) &= P_{\geq n}(\sum_{i=0}^{n} \sum_{k=0}^{N} a_i \partial^i u_k \partial^k \ ) = P_{\geq n}(\sum_{i=0}^{n} \sum_{k=0}^{N} \sum_{j=0}^{i} \begin{pmatrix} i \\ j \end{pmatrix} a_i u_k^{(j)} \partial^{i+k-j} \ ) \\
&= \sum_{j=0}^{N} \ (\sum_{l=0}^{j} \sum_{k=K(l,n)}^{l} \begin{pmatrix} n+k-l \\ j-l \end{pmatrix} a_{n+k-l} u_{N-k}^{(j-l)} \ ) \ \partial^{N+n-j} \ .
\end{aligned}
\tag{6.19}
$$

Equating the differential orders of (6.18) and (6.19) one finds the recursive relations (6.16). This finishes the proof of Theorem 6.

We remark that the recursion (6.16) does indeed define the new coefficients $\tilde{u}_i$ in terms of the old coefficients u_i and the eigenfunctions $\chi_1, .., \chi_n$ (via $a_0, .., a_{n-1}$). We observe that (6.16) is of the form

$$\tilde{u}_N = u_N \quad , \quad \tilde{u}_{N-j} = \mathcal{G}_j(u_{N-j}, .., u_N) - \mathcal{H}_j(\tilde{u}_{N-j+1}, .., \tilde{u}_N) \quad , \quad j = 1, .., N \quad , \tag{6.20}$$

with the differential expressions $\mathcal{G}_j$, $\mathcal{H}_j$ given by (6.16). Hence, for any given j the $\tilde{u}$'s in the right hand side of (6.20) are known in terms of the u's from the relations for the lower values of j. In particular, in view of (5.8) we assume that the highest coefficient is given by a constant matrix $u_N = c$ and find

$$\begin{aligned} \tilde{u}_N &= u_N = c \ , \\ \tilde{u}_{N-1} &= u_{N-1} + [a_{n-1}, c] \ , \\ \tilde{u}_{N-2} &= u_{N-2} + [a_{n-1}, u_{N-1}] + [a_{n-2}, c] + n u_{N-1,x} - N c a_{n-1,x} + N[c, a_{n-1}]a_{n-1} \end{aligned} \tag{6.21}$$

for the highest coefficients. For the special choice $c = 1$ we also assume $u_{N-1} = [c, w_1] = 0$ (4.9) and find

$$\tilde{u}_N = 1 \quad , \quad \tilde{u}_{N-1} = 0 \quad , \quad \tilde{u}_{N-2} = u_{N-2} - N a_{n-1,x} \quad . \tag{6.22}$$

We remark that in the case of the scalar KP hierarchy, where all coefficients and (adjoint) eigenfunctions are scalar functions, the coefficients a_i of the gauge operator are given by Wronskian expressions (3.25). Hence, in the scalar case the coefficients of the Darboux transformed operator are given in terms of the original coefficients and various Wronskian determinants of the eigenfunctions $\chi_1, .., \chi_n$ used for the transformation. In particular, with (6.22) and (3.26), for scalar Lax operators of the form $L = \partial^N + u_{N-2}\partial^{N-2} + .. + u_0 + \sum_i \Phi_i \partial^{-1} \Psi_i^\dagger$ the highest nontrivial coefficient u_{N-2} is transformed into

$$u_{N-2} \quad \rightarrow \quad \tilde{u}_{N-2} = u_{N-2} + N \frac{d^2}{dx^2} ln\ \mathcal{W}(\chi_1, .., \chi_n) \tag{6.23}$$

by the iterated DT. Similar formulas involving Wronskian determinants can be obtained from (6.16) for all coefficients $\tilde{u}_{N-2}, .., \tilde{u}_0$.

7 Examples

To illustrate the previous theorems we will apply the results to the reduction of the KP hierarchy associated with the Lax operator

$$L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial + \begin{pmatrix} 0 & r \\ q & 0 \end{pmatrix} + \sum_{i=1}^{m} \Phi_i \partial^{-1} \Psi_i^\dagger \tag{7.1}$$

with vector valued Φ_i, Ψ_i. We will show that the simplest nonlinear evolution equations associated with (7.1) are the following "nonlinearized" versions of flows of the NLS hierarchy

$$U_y = 2\sigma_3 \sum_{i=1}^{m} (\Phi_i \Psi_i^\dagger)^{off} \quad , \quad \Phi_{iy} = \sigma_3 \Phi_{ix} + U\Phi_i \quad , \quad \Psi_{iy} = \sigma_3 \Psi_{ix} - U^\dagger \Psi_i \tag{7.2}$$

and

$$
\begin{aligned}
U_t =& \ \tfrac{1}{2}\sigma_3 U_{xx} + \sigma_3 U^3 + [\textstyle\sum_{j=1}^m (\Phi_j \Psi_j^\dagger)^{diag}, U] + \sigma_3 \sum_{j=1}^m (\Phi_{jx}\Psi_j^\dagger - \Phi_j \Psi_{jx}^\dagger)^{off} \\
\Phi_{it} =& \ \sigma_3 \Phi_{ixx} + U\Phi_{ix} + \tfrac{1}{2}U_x \Phi_i + \tfrac{1}{2}\sigma_3 U^2 \Phi_i + \textstyle\sum_{j=1}^m \Phi_j \Psi_j^\dagger \Phi_i + \sum_{j=1}^m (\Phi_j \Psi_j^\dagger)^{diag} \Phi_i \quad , \\
\Psi_{it} =& -\sigma_3 \Psi_{ixx} + U^\dagger \Psi_{ix} + \tfrac{1}{2}U_x^\dagger \Psi_i - \tfrac{1}{2}\sigma_3 U^2 \Psi_i - \textstyle\sum_{j=1}^m \Psi_j \Phi_j^\dagger \Psi_i - \sum_{j=1}^m (\Psi_j \Phi_j^\dagger)^{diag} \Psi_i \quad .
\end{aligned}
\tag{7.3}
$$

Here

$$
U = \begin{pmatrix} 0 & r \\ q & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7.4}
$$

and the superscripts off and $diag$ refer to a decomposition $a = a^{diag} + a^{off}$ of a matrix a into a diagonal and an off-diagonal part. The above flows occur as constraints of the 2+1-dimensional integrable dynamics encoded in the KP hierarchy. As a first step we derive the simplest 2+1-dimensional system which can be extracted from the two-component KP hierarchy (4.4). Then we will use a symmetry constraint involving the symmetry of Theorem 3 generated by squared eigenfunctions to obtain the reduced equations (7.2) and (7.3). Finally we will derive Wronskian solution formulas for these systems from the Darboux formalism of the previous sections.

We restrict the considerations of the previous sections to matrices of the dimension 2×2 . We consider the evolutions (4.4) of the KP hierarchy associated with the operators $A = \sigma_3\partial$ and $B = \sigma_3\partial^2$. The associated times are to be called $t_A = y$ and $t_B = t$. With (4.1) one calculates

$$
\begin{aligned}
W\sigma_3\partial W^{-1} &= \sigma_3\partial + [w_1,\sigma_3] + ([w_2,\sigma_3] - \sigma_3 w_{1x} - [w_1,\sigma_3]w_1)\partial^{-1} + (..)\partial^{-2} + .. \\
W\sigma_3\partial^2 W^{-1} &= \sigma_3\partial^2 + [w_1,\sigma_3]\partial + [w_2,\sigma_3] - 2\sigma_3 w_{1x} - [w_1,\sigma_3]w_1 + (..)\partial^{-1} + ..
\end{aligned}
\tag{7.5}
$$

and obtains

$$
w_{1y} = [\sigma_3, w_2] + \sigma_3 w_{1x} + [w_1,\sigma_3]w_1 \tag{7.6}
$$

from the highest order of $W_y = -P_{<0}(W\sigma_3\partial W^{-1})W$. Hence, we find the following parametrization of the differential operators

$$
\begin{aligned}
M_A &= P_{\geq 0}(W\sigma_3\partial W^{-1}) = \sigma_3\partial + [w_1,\sigma_3] \quad , \\
M_B &= P_{\geq 0}(W\sigma_3\partial^2 W^{-1}) = \sigma_3\partial^2 + [w_1,\sigma_3]\partial - w_{1y} - \sigma_3 w_{1x}
\end{aligned}
\tag{7.7}
$$

in terms of the field w_1. We introduce

$$
U = \begin{pmatrix} 0 & r \\ q & 0 \end{pmatrix} = [w_1,\sigma_3] = -2\sigma_3 w_1^{off} \quad , \quad D = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = -2w_1^{diag} \quad , \tag{7.8}
$$

so that we can decompose w_1 into its off-diagonal and its diagonal part via $w_1 = w_1^{off} + w_1^{diag} = -\frac{1}{2}(\sigma_3 U + D)$. The zero-curvature condition $M_{At} - M_{By} = [M_B, M_A]$ then leads to the evolution equation

$$
U_t = \frac{1}{2}\sigma_3(U_{xx} + U_{yy}) + \frac{1}{2}\sigma_3(UD_x + D_x U) + \frac{1}{2}[D_y, U] + \frac{1}{2}(D_{yy} - D_{xx} + (U^2)_x + \sigma_3(U^2)_y) \quad . \tag{7.9}
$$

From the diagonal part of (7.6) one has

$$\sigma_3 D_y - D_x + U^2 = 0 \quad , \tag{7.10}$$

so that $D_{yy} - D_{xx} + (U^2)_x + \sigma_3(U^2)_y = 0$ and the diagonal part of (7.9) is satisfied identically. The combined system (7.9)/(7.10) may be rewritten as

$$U_t = \frac{1}{2}\sigma_3(U_{xx} + U_{yy}) + \sigma_3 U^3 + [D_y, U] \quad , \quad \sigma_3 D_y - D_x + U^2 = 0 \quad , \tag{7.11}$$

i.e.

$$\begin{array}{rclrcl} r_t & = & \frac{1}{2}(r_{xx} + r_{yy}) + r^2 q + (a_y - b_y) r \quad , & a_y - a_x + rq & = & 0 \quad , \\ q_t & = & -\frac{1}{2}(q_{xx} + q_{yy}) - q^2 r - (a_y - b_y) q \quad , & b_y + b_x - rq & = & 0 \quad , \end{array} \tag{7.12}$$

which represents as 2+1 dimensional version of the AKNS system. As we have $(a+b)_y = (a-b)_x$ we may introduce the field ϕ by $a + b = \phi_x$, $a - b = \phi_y$, which has to satisfy $\phi_{xx} - \phi_{yy} = 2rq$. If we replace U_t by iU_t and impose the constraints $q = \bar{r}$, $\phi = \bar{\phi}$ (the bar meaning complex conjugation), then this systems reduces to the Davey-Stewartson equation

$$ir_t = \frac{1}{2}(r_{xx} + r_{yy}) + |r|^2 r + \phi_{yy} r \quad , \quad \phi_{xx} - \phi_{yy} = 2|r|^2 \ . \tag{7.13}$$

Exact solutions of the AKNS system (7.11)/(7.12) can be obtained easily from the DT of Theorem 3 and its iterated version. For the differential operators

$$\begin{array}{rcl} M_A & = & \sigma_3 \partial + U \quad , \\ M_B & = & \sigma_3 \partial^2 + U\partial + \frac{1}{2}U_x + \frac{1}{2}\sigma_3 U^2 + \frac{1}{2}\sigma_3 U_y + D_y \end{array} \tag{7.14}$$

obtained from (7.7) and (7.10) we consider matrix eigenfunctions $\chi_1, .., \chi_n$ satisfying the scattering problems

$$\begin{array}{rclcl} \chi_y & = & P_0(M_A \chi) & = & \sigma_3 \chi_x + U\chi \quad , \\ \chi_t & = & P_0(M_B \chi) & = & \sigma_3 \chi_{xx} + U\chi_x + \frac{1}{2}U_x \chi + \frac{1}{2}\sigma_3 U^2 \chi + \frac{1}{2}\sigma_3 U_y \chi + D_y \chi \quad . \end{array} \tag{7.15}$$

Let $T = \partial^n + a_{n-1}\partial^{n-1} + .. + a_0$ be the transformation operator of Theorem 2, where the matrices $a_0, .., a_{n-1}$ are given as the solutions of the algebraic system

$$\sum_{i=0}^{n} a_i \chi_j^{(i)} \ = \ 0 \quad , \quad j = 1, .., n \quad , \quad a_n = 1 \quad . \tag{7.16}$$

With (4.22) one obtains $\tilde{w}_1 = w_1 + a_{n-1}$ as the leading order coefficient for the transformed dressing operator, so that the DT leads to

$$\tilde{U} = [\tilde{w}_1, \sigma_3] = U - 2\sigma_3 a_{n-1}^{off} \quad , \quad \tilde{D} = -2\tilde{w}_1^{diag} = D - 2a_{n-1}^{diag} \tag{7.17}$$

as the new solutions of the AKNS system (7.11). In particular, if we start with the trivial solution $W = 1$ of the KP hierarchy (i.e. $U = D = 0$), we use n independent solutions $\chi_1, .., \chi_n$ of the scattering problems

$$\chi_y = \sigma_3 \chi_x \quad , \quad \chi_t = \sigma_3 \chi_{xx} \quad , \tag{7.18}$$

and obtain the nontrivial solution

$$\tilde{U} = -2\sigma_3 a_{n-1}^{off} \quad , \quad \tilde{D} = -2a_{n-1}^{diag} \tag{7.19}$$

for (7.11) after computation of a_{n-1} from the algebraic equations (7.16). The equations (7.19) provide a simple Wronskian formulation for exact solutions of the 2+1-dimensional AKNS system (7.11).

We will now impose a constraint on the AKNS system leading to the integrable hierarchy associated with the Lax operator (7.1). We start with eigenfunctions Φ_i and adjoint eigenfunctions Ψ_i satisfying the evolutions

$$\begin{aligned} \Phi_{iy} &= P_0(M_A\Phi) \quad , \quad \Psi_{iy} = -P_0(M_A^*\Psi) \quad , \\ \Phi_{it} &= P_0(M_B\Phi) \quad , \quad \Psi_{it} = -P_0(M_B^*\Psi) \quad . \end{aligned} \tag{7.20}$$

According to Theorem 3 the squared eigenfunctions generate a symmetry of the KP hierarchy generated by the evolution equation

$$W_z = -\sum_{i=1}^m \Phi_i\partial^{-1}\Psi_i^\dagger \; W = -(\sum_{i=1}^m \Phi_i\Psi_i^\dagger\partial^{-1} + (..)\partial^{-2} + ..)(1 + w_1\partial^{-1} + ..) = -\sum_{i=1}^m \Phi_i\Psi_i^\dagger \,\partial^{-1} + .. \; , \tag{7.21}$$

from which we obtain in particular

$$w_{1z} = -\sum_{i=1}^m \Phi_i\Psi_i^\dagger \tag{7.22}$$

and hence

$$U_z = [w_{1z}, \sigma_3] = 2\sigma_3 \sum_{i=1}^m (\Phi_i\Psi_i^\dagger)^{off} \quad , \quad D_z = -2w_{1z}^{diag} = 2\sum_{i=1}^m (\Phi_i\Psi_i^\dagger)^{diag} \quad . \tag{7.23}$$

We impose the constraint $P_{<0}(WAW^{-1}) = \sum_i \Phi_i\partial^{-1}\Psi_i^\dagger$, which coincides with the symmetry reduction $W_z = W_y$. We choose the Lax operator

$$L = W\sigma_3\partial W^{-1} = \sigma_3\partial + U + (..)\partial^{-1} + .., \tag{7.24}$$

which in this reduction assumes the form (7.1). The y-evolution is associated with the Lax equation

$$L_y = [M_A, L] = -[\sum_{i=1}^m \Phi_i\partial^{-1}\Psi_i^\dagger \, , L] = L_z \quad , \tag{7.25}$$

which implies the dynamics (7.2). The next equation in the constrained hierarchy is given by the Lax equation $L_t = [M_B, L]$. With (7.14) and $U_y = U_z,\ D_y = D_z$ given by (7.23) one finds

$$M_B = \sigma_3\partial^2 + U\partial + \frac{1}{2}U_x + \frac{1}{2}\sigma_3 U^2 + \sum_{i=1}^m \Phi_i\Psi_i^\dagger + \sum_{i=1}^m (\Phi_i\Psi_i^\dagger)^{diag} \quad . \tag{7.26}$$

The Lax equation $L_t = [M, L]$ given by (7.1) and (7.26) yields the dynamics (7.3).

Higher symmetries of (7.2)/(7.3) can be obtained from the higher KP equations (4.4) (with any undressed operator A) using the constraints implied by $P_{<0}(W\sigma_3\partial W^{-1}) = \sum_i \Phi_i\partial^{-1}\Psi_i^\dagger$.

Finally, we will use the DT of the last sections to derive solution formulas for the constrained KP hierarchy. First, we convert the scattering problem associated with the pseudo-differential Lax operator (7.1) into an ordinary differential problem using Lemma 3. We find that (7.2) and (7.3) are given as the compatibility conditions of the spectral problem

$$\sigma_3\chi_x + U\chi + \sum_{i=1}^{m}\Phi_i S_i = \chi\,\Lambda \quad , \quad S_{ix} = \Psi_i^\dagger\chi \quad , \tag{7.27}$$

with

$$\chi_y = \sigma_3\chi_x + U\chi \quad , \quad S_{iy} = res(\partial^{-1}\Psi_i^\dagger(\sigma_3\partial + U)\chi\partial^{-1}) = \Psi_i^\dagger\sigma_3\chi \tag{7.28}$$

and

$$\chi_t = P_0(M_B\chi) \quad , \quad S_{it} = res(\partial^{-1}\Psi_i^\dagger M_B\chi\partial^{-1}) = \Psi_i^\dagger\sigma_3\chi_x - \Psi_{ix}^\dagger\sigma_3\chi \quad , \tag{7.29}$$

respectively. Here we consider matrix solutions χ, so that the potentials $S_i = S(\Psi_i, \chi)$ are row vectors. In order to use Theorem 5 we need a seed solution of the system (7.2) or (7.3), respectively. We choose $U = 0$, $\Psi_i = 0$ and $\Phi_i = (\alpha_i, \beta_i)^\dagger$ with constants α_i, β_i. We consider solutions χ of the above equations, where the associated potentials $S_i = S(\Psi_i, \chi)$ are arbitrary constants. Hence, we have to solve the spectral problem

$$\sigma_3\chi_x + C = \chi\Lambda \tag{7.30}$$

with a constant matrix

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \sum_{i=1}^{m}\begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} S_i \quad . \tag{7.31}$$

We specialize the spectral matrix $\Lambda = diag(\lambda, \mu)$ to diagonal form. The evolution equations simplify to

$$\chi_y = \sigma_3\chi_x \quad , \quad \chi_t = \sigma_3\chi_{xx} \quad . \tag{7.32}$$

The solutions of (7.30) and (7.32) are given by

$$\chi = \begin{pmatrix} \frac{c_{11}}{\lambda} + d_{11}e^{\lambda x + \lambda y + \lambda^2 t} & \frac{c_{12}}{\mu} + d_{12}e^{\mu x + \mu y + \mu^2 t} \\ \frac{c_{21}}{\lambda} + d_{21}e^{\lambda x - \lambda y - \lambda^2 t} & \frac{c_{22}}{\mu} + d_{22}e^{\mu x - \mu y - \mu^2 t} \end{pmatrix} \tag{7.33}$$

with arbitrary constants d_{ij}. The constants should be chosen such that the matrix χ is invertible for all values of x, y and t. According to Theorem 5/Theorem 6 and (6.21), i.e. $\tilde{U} = U + [a_{n-1}, \sigma_3]$ with $a_{n-1} = a_0 = -\chi_x\chi^{-1}$, the DT

$$\begin{aligned} U = 0 \qquad & \rightarrow \quad \tilde{U} = -[\chi_x\chi^{-1}, \sigma_3] = 2\sigma_3(\chi_x\chi^{-1})^{off} \quad , \\ \Phi_i = \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} \quad & \rightarrow \quad \tilde{\Phi}_i = -\chi_x\chi^{-1}\begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} \quad , \\ \Psi_i = 0 \qquad & \rightarrow \quad \tilde{\Psi}_i^\dagger = -S_i\chi^{-1} \end{aligned} \tag{7.34}$$

with χ given by (7.33) leads to a special solution $(\tilde{U}, \tilde{\Phi}_i, \tilde{\Psi}_i)$ of the (2+1)-dimensional AKNS system (7.11)/(7.12). For fixed value of t equation (7.2) is satisfied, for fixed y a solution of (7.3) is provided. Further solutions are obtained by taking several copies $\chi_1, .., \chi_n$ of (7.33), i.e.

$$\chi_j = \begin{pmatrix} \frac{c_{j,11}}{\lambda_j} + d_{j,11} e^{\lambda_j x + \lambda_j y + \lambda_j^2 t} & \frac{c_{j,12}}{\mu_j} + d_{j,12} e^{\mu_j x + \mu_j y + \mu_j^2 t} \\ \frac{c_{j,21}}{\lambda_j} + d_{j,21} e^{\lambda_j x - \lambda_j y - \lambda_j^2 t} & \frac{c_{j,22}}{\mu_j} + d_{j,22} e^{\mu_j x - \mu_j y - \mu_j^2 t} \end{pmatrix} \tag{7.35}$$

with

$$\begin{pmatrix} c_{j,11} & c_{j,12} \\ c_{j,21} & c_{j,22} \end{pmatrix} = \sum_{i=1}^{m} \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} S_{ij}. \tag{7.36}$$

The potentials $S_{ij} = S(\Psi_i, \chi_j) = S(0, \chi_j)$ are arbitrary constant row vectors. One computes the matrices $a_0, .., a_{n-1}$ and $b_1, .., b_n$ from the equations (3.21) and (3.22) and with Theorem 5 and (6.21) the transformations

$$\begin{aligned} U = 0 \quad & \rightarrow \quad \tilde{U} = [a_{n-1}, \sigma_3] = -2\sigma_3 a_{n-1}^{off} \quad , \\ \Phi_i = \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} \quad & \rightarrow \quad \tilde{\Phi}_i = a_0 \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} \quad , \\ \Psi_i = 0 \quad & \rightarrow \quad \tilde{\Psi}_i^\dagger = -\sum_{j=1}^{n} S_{ij} b_j \end{aligned} \tag{7.37}$$

yield nontrivial solutions $\tilde{U}, \tilde{\Phi}_i, \tilde{\Psi}_i$ of the flows (7.2) and (7.3). The final result (7.37) represents the Wronskian solution formula for the constrained KP flows of the nonlinearized NLS hierarchy.

8 Conclusions and perspectives

We have given a systematic approach to the symmetry constraint of the KP hierarchy as proposed recently by various authors ([16-21,29-32]. It was shown that pairs of eigenfunctions and adjoint eigenfunctions generate a symmetry of the soliton systems via Lax equations involving the pseudo-differential operator $\sum_i \Phi_i \partial^{-1} \Psi_i^\dagger$. On the operator level this is the expression corresponding to squared eigenfunction. In this version, this symmetry can be considered for the general multicomponent KP hierarchy. Symmetry reductions are shown to be naturally associated with pseudo-differential Lax operators of the type (1.1). The associated integrable equations represent the nonlinearized versions of the soliton systems associated with differential Lax representations, which are obtained in the reduction Φ_i (or Ψ_i)$\rightarrow 0$.

Following the general concept of DTs we have adapted this formalism to include pseudo-differential operators. As a result, the DT for all operators of the type (1.1) is established and can be used to generate exact solutions of the associated nonlinearized soliton equations. As an example we have demonstrated the procedure for the nonlinearized NLS system in 1+1 dimensions, which is obtained as a symmetry reduction of the 2+1-dimensional AKNS hierarchy. Wronskian type solution formulas have been derived.

In fact, the Darboux formalism used here holds for arbitrary nonlinear equations either encoded in a Lax equation $L_t = [M, L]$ or in the zero-curvature formulation as the compatibility condition for two linear problems $\Phi_{t_A} = P_0(M_A\Phi)$, $\Phi_{t_B} = P_0(M_B\Phi)$. It has been demonstrated that, even if all operators involved are differential operators, the use of pseudo-differential integration symbols is an extremely convenient tool. In this terminology a DT is given as a simple gauge transformation

$$L \to \chi\partial\chi^{-1}\ L\ \chi\partial^{-1}\chi^{-1} \quad , \quad M \to P_{\geq 0}(\ \chi\partial\chi^{-1}\ M\ \chi\partial^{-1}\chi^{-1}\ ) \tag{8.1}$$

involving an eigenfunction χ. In this formulation no assumptions on the particular form of the operators are needed to establish the Darboux theorem. The class of problems arising from the multicomponent KP hierarchy has the additional feature that a canonical parametrization of the operators in terms of a dressing operator is given. This parametrization is preserved under the DT, as for the KP hierarchy the general DT (8.1) can be traced back to a Darboux type transformation of the dressing operator. As a result, the form invariance of the Lax pairs grants an invariance of the nonlinear equations under the DT.

It should be mentioned, however, that this argument on the invariance of the parametrizations will only apply in the situation when a dressing operator $W = 1 + \sum_j w_j\partial^{-j}$ with matrix coefficients w_j in general position is used. A number of interesting integrable systems, however, reside in the general scenario as special reductions. As a typical example, the Davey-Stewartson system (7.13) may be characterized as a reduction of the AKNS system (7.11) by imposing a constraint of the type $U = \pm U^\dagger$ (hermitian transposition). In terms of the KP hierarchy, this may be interpreted as a reduction to unitary dressing operators satisfying $WW^* = 1$. The DTs as presented here will in general <u>not</u> preserve such a constraint. As a result, the general DT cannot be used immediately to generate exact solutions of these systems, and certain adaptations to the general Darboux picture have to be made. In particular cases additional conditions on the eigenfunctions used for the transformation may be imposed, so that the DT will leave the algebraic conditions invariant. A more general approach is given by the notion of binary DTs as explained by Matveev and Salle ([6]), where not an eigenfunction, but a pair of eigenfunctions and adjoint eigenfunctions is used to trigger the transformation. The associated potentials of Lemma 3 are crucial in this case and enter the formulas explicitly. The binary DTs will preserve algebraic constraints between the scattering operators and their adjoints, when corresponding relations between the triggering eigenfunctions and adjoint eigenfunctions are imposed. In particular, the dromion solution of the Davey-Stewartson equation may be generated this way. Again, it turns out that the use of pseudo-differential symbols is very useful to give a general characterization of this type of transformation. A detailed analysis is to be published elsewhere.

Another line of future research should include the development of a general picture for Darboux and binary DTs for the Miura-modified problems. For the scalar case a general concept of DTs, Miura transformations and reciprocal links in pseudo-differential terminology was proposed in Ref. [7]. The results encapsulate the modified KP hierarchy and the Dym hierarchy and their connection with the KP hierarchy. Generalizations to the multicomponent case as well as a corresponding concept of binary DTs are still to be developed.

Another interesting aspect of integrable systems is given by their Hamiltonian features. In Ref. [32] it was shown that for the scalar KP hierarchy there is a natural bi-Hamiltonian structure, which involves the scattering operators as well as pairs of eigenfunctions and adjoint eigenfunctions. These Poisson brackets are extended versions of the usual linear and quadratic brackets of Lie-Poisson and Gelfand-Dikii type. The extended brackets seem to provide a natural scenario to study the Hamiltonian properties of DTs and binary DTs.

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Appendix: Proof of Theorem 2

We first show that $\tilde{L}, \tilde{M}$ is obtained as a gauge transformation of L, M with the transformation operator given by (3.23). From the recursive construction it is clear that T_n is given by

$$T_n \;=\; (\partial - (\chi_n[n-1])_x (\chi_n[n-1])^{-1})\, (\partial - (\chi_{n-1}[n-2])_x (\chi_{n-1}[n-2])^{-1})\, ..(\partial - (\chi_1[0])_x (\chi_1[0])^{-1}) \;\; . \quad \text{(A.1)}$$

Hence, T_n is an n-th order differential operator with a normalized leading coefficient which annihilates all functions $\chi_1, .., \chi_n$:

$$P_0(T_n\chi_i) = P_0(\; (\partial - (\chi_n[n-1])_x (\chi_n[n-1])^{-1}) \; .. \; (\partial - (\chi_i[i-1])_x (\chi_i[i-1])^{-1})\; \chi_i[i-1] \;) \;=\; 0 \;\; . \quad \text{(A.2)}$$

Using the ansatz $T_n = \partial^n + \sum_{i=0}^{n-1} a_i \partial^i$ the conditions $P_0(T_n\chi_j) = 0$ with $j = 1, .., n$ leads to the n matrix-valued algebraic equations (3.21), by which the n matrix-valued coefficients $a_0, .., a_{n-1}$ are determined in terms of $\chi_1, .., \chi_n$. We now show that T_n^{-1} given by (3.23) is indeed the inverse of T_n. For this we observe that $T_n T_n^{-1}$ yields a purely differential operator, as

$$P_{<0}(T_n T_n^{-1}) = P_{<0}(\sum_{i=0}^{n}\sum_{j=1}^{n} a_i \partial^i \chi_j \partial^{-1} b_j) = \sum_{j=1}^{n}(\sum_{i=0}^{n} P_0(a_i \partial^i \chi_j)\;)\partial^{-1} b_j = 0 \quad \text{(A.3)}$$

due to (3.21). On the other hand, we may use

$$\chi_j \partial^{-1} b_j = \sum_{i\geq 0} \partial^{-1-i} \chi_j^{(i)} b_j \quad \text{(A.4)}$$

and (3.22) to rewrite

$$T_n^{-1} = \sum_{i\geq 0} \partial^{-1-i} \sum_{j=1}^{n} \chi_j^{(i)} b_j = \partial^{-n} + \sum_{i\geq n} \partial^{-1-i} \sum_{j=1}^{n} \chi_j^{(i)} b_j = \partial^{-n} + \; \textit{lower orders} \; . \quad \text{(A.5)}$$

Hence, $T_n T_n^{-1} = 1 + \textit{lower orders} = 1$, as no negative differential orders can occur due to the previous argument.

Concerning the transformation (3.24) of (adjoint) eigenfunctions we observe that the transformation formula for $\tilde{\Phi}$ is obtained trivially, as one finds $\tilde{\Phi} = P_0(T_n\Phi)$ from the recursion (3.20) . The formula for the $\tilde{\Psi}$ is easily guessed by applying the adjoint of T_n^{-1} to the original Ψ and formally replacing terms such as $b_j^\dagger \partial^{-1} \chi_j^\dagger \Psi$ by $b_j^\dagger S(\Psi_i, \chi)^\dagger$. However, a rigorous argument is required to show that this formula coincides with the results of the recursion (3.20). We note that due to the superposition formula (3.20.vi) all potentials $S_j[k] = S(\Psi[k], \chi_j[k])$ are determined by the potentials $S_j[0] = S_j = S(\Psi, \chi_j)$. We use (3.20.v) and (3.20.vi) to find

$$\begin{aligned} -\Psi[k]^\dagger \chi_k[k-1] = S_k[k-1] \;\; &= \Psi[k-1]^\dagger \chi_k[k-2] + S_k[k-2] \\ &= \Psi[k-1]^\dagger \chi_k[k-2] + \Psi[k-2]^\dagger \chi_k[k-3] + S_k[k-3] \\ &= .. \\ &= \textstyle\sum_{j=1}^{k-1} \;\; \Psi[j]^\dagger \chi_k[j-1] \;+\; S_k[0] \;\; . \end{aligned} \quad \text{(A.6)}$$

With $k=1,..,n$ we combine the equations $\sum_{j=1}^{k}\Psi[j]^{\dagger}\chi_k[j-1]=-S_k[0]$ to the following system

$$\begin{pmatrix}\Psi[1]\\ \Psi[2]\\ ..\\ \Psi[n]\end{pmatrix}^{\dagger}\begin{pmatrix}\chi_1[0] & \chi_2[0] & .. & \chi_n[0]\\ 0 & \chi_2[1] & .. & \chi_n[1]\\ .. & .. & .. & ..\\ 0 & .. & 0 & \chi_n[n-1]\end{pmatrix}=\begin{pmatrix}-S_1[0]\\ -S_2[0]\\ ..\\ -S_n[0]\end{pmatrix}^{\dagger} \quad . \tag{A.7}$$

Introducing the inverse (b_{jk}) of the matrix $(\chi_i[j])$ all $\Psi[k]$ can be expressed in terms of $S_j=S_j[0]$ and $\chi_j[i]$ via

$$\Psi[k]=-\sum_{j=1}^{k}S_j\ b_{jk}\quad ,\quad k=1,..,n\quad , \tag{A.8}$$

where the elements b_{jk} of the inverse matrix are defined as the solution of the equations

$$\sum_{j=i}^{k}\chi_j[i-1]b_{jk}=\delta_{ik}\quad ,\quad i,k=1,..,n\quad . \tag{A.9}$$

In particular, for $k=n$ we introduce $b_j=b_{jn}$ and obtain

$$\Psi[n]=-\sum_{j=1}^{n}S_j\ b_j\quad , \tag{A.10}$$

where the matrix-valued coefficients b_j are determined by the n matrix-valued equations

$$\sum_{j=i}^{n}\chi_j[i-1]b_j=\delta_{in}\quad ,\quad i=1,..,n\quad . \tag{A.11}$$

Equation (A.10) represents the formula (3.24.ii), where we still have to show that the system (A.11) determining the b_j's is equivalent to the system (3.22) which involves the Wronskian matrix of the original eigenfunctions $\chi_1,..,\chi_n$. For this we first note that there are coefficients α_{ik} (independent of j) such that we can express all $\chi_i[j]$ in terms of the original function χ_j and its derivatives, i.e.

$$\chi_j[i]=P_0(\ (\partial-(\chi_i[i-1])_x(\chi_i[i-1])^{-1})\ ..(\partial-(\chi_1[0])_x(\chi_1[0])^{-1})\chi_j\ )\ =\ \chi_j^{(i)}\ +\ \sum_{k=0}^{i-1}\alpha_{ik}\chi_j^{(k)}\quad . \tag{A.12}$$

In matrix form these relations can be written as

$$\begin{pmatrix}\chi_1[0] & \chi_2[0] & .. & \chi_n[0]\\ 0 & \chi_2[1] & .. & \chi_n[1]\\ .. & .. & .. & ..\\ 0 & .. & 0 & \chi_n[n-1]\end{pmatrix}=\begin{pmatrix}1 & 0 & .. & 0\\ \alpha_{21} & 1 & .. & 0\\ .. & .. & .. & ..\\ \alpha_{n1} & .. & \alpha_{n,n-1} & 1\end{pmatrix}\begin{pmatrix}\chi_1 & \chi_2 & .. & \chi_n\\ \chi_{1x} & \chi_{2x} & .. & \chi_{nx}\\ .. & .. & .. & ..\\ \chi_1^{(n-1)} & \chi_2^{(n-1)} & .. & \chi_n^{(n-1)}\end{pmatrix}\quad , \tag{A.13}$$

i.e. the matrix $(\chi_j[i])$ provides a factorization of the Wronskian matrix of $\chi_1,..,\chi_n$ into lower and upper triangular matrices. Starting with (A.11), i.e.

$$\begin{pmatrix}\chi_1[0] & \chi_2[0] & .. & \chi_n[0]\\ 0 & \chi_2[1] & .. & \chi_n[1]\\ .. & .. & .. & ..\\ 0 & .. & 0 & \chi_n[n-1]\end{pmatrix}\begin{pmatrix}b_1\\ b_2\\ ..\\ b_n\end{pmatrix}=\begin{pmatrix}0\\ ..\\ 0\\ 1\end{pmatrix}\quad , \tag{A.14}$$

we multiply from the left with the inverse of the lower triangular matrix in (A.13). Thus one finds the equivalent system (3.22) determining the coefficients b_j in terms of $\chi_1, .., \chi_n$ and their derivatives. This finishes the proof of Theorem 2.