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Wronskian solutions of the constrained KP hierarchy

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Abstract

The integrable Kadomtsev-Petviashvili (KP) hierarchy is compatible with generalized k-constraints of the type $(L^k)_- = \sum_i q_i \, \partial_x^{-1} r_i$. A large class of solutions – among them solitons – can be represented by Wronskian determinants of functions satisfying a set of linear equations. In this paper we shall obtain additional conditions for these functions imposed by the constraints.

I The constrained KP hierarchy

The dynamics of the KP hierarchy is defined by the Lax equations

$$L_{t_n} = [L, (L^n)_-] \,, \tag{I.1}$$

with the micro-differential operator $L = \partial_x + u_2\partial_x^{-1} + u_3\partial_x^{-2} + \cdots$. The dressing operator $W = 1 + w_1\partial_x^{-1} + w_2\partial_x^{-2} + \cdots$, related to the Lax operator by $L = W\partial_x W^{-1}$, satisfies Sato's equations

$$W_{t_n} = -(W\partial_x^n W^{-1})_- W \,. \tag{I.2}$$

The essence of Sato's theory [1–5] is the fact that solutions of Sato's equations can be expressed by a single function τ, which has to satisfy a hierarchy

*Permanent address: FB 17-Mathematik, University of Paderborn, 33095 Paderborn, Germany of *bilinear* equations. It is well-known [6–11] that the KP dynamics (I.1) is compatible with generalized k-constraints of the form

$$(L^k)_- \;=\; \sum_{i=1}^{M} q_i\, \partial_x^{-1}\, r_i \tag{I.3}$$

with suitable functions q_i and r_i. We note that such Lax operators are characterized by the existence of an M-th order differential operator

$$A \;=\; \partial_x^M + a_{M-1}\,\partial_x^{M-1} + \cdots + a_0$$

such that AL^k is a differential operator. The factor A needs to annihilate the functions q_i, i.e. $A(q_1) = \ldots = A(q_N) = 0$. Since an operator may be reconstructed from its kernel, A is uniquely determined if the Wronskian determinant $\mathcal{W}(q_1, \ldots, q_N)$ does not vanish. Similarly, there exists

$$B \;=\; \partial_x^M + b_{M-1}\,\partial_x^{M-1} + \cdots + b_0$$

such that L^kB is a differential operator. The functions r_i span the kernel of the adjoint B^*. The constraint (I.3) reduces the KP dynamics to the hierarchy

$$(L^k)_{t_n} = [L^k, (L^n)_-]\,, \quad (q_i)_{t_n} = (L^n)_+(q_i)\,, \quad (r_i)_{t_n} = -(L^{*n})_+(r_i)$$

of $1+1$-dimensional integrable equations For $M = 1$ the first three constraints yield the the AKNS hierarchy ($k = 1$), the Yajima-Oikawa hierarchy ($k = 2$) and a coupled Boussinesq type ($k = 3$) hierarchy [6–11].

II Wronskian solutions

In the following we consider linearly independent functions $f_1, \ldots, f_N$ satisfying the infinite set of linear equations

$$\frac{\partial f_i}{\partial t_n} \;=\; f_i^{(n)} \equiv \frac{\partial^n f_i}{\partial x^n} \tag{II.1}$$

Their Wronskian determinant

$$\mathcal{W}(f_1, \ldots, f_N) \;=\; \begin{vmatrix} f_1^{(0)} & \ldots & f_1^{(N-1)} \\ \vdots & \ddots & \vdots \\ f_N^{(0)} & \ldots & f_N^{(N-1)} \end{vmatrix}$$

is a τ-function of the KP hierarchy [4]. This implies that the N-th order differential operator $W = \partial_x^N + w_1 \partial_x^{N-1} + \cdots + w_N$ given by

$$W = \frac{1}{\mathcal{W}(f_1, \ldots, f_N)} \begin{vmatrix} f_1^{(0)} & f_1^{(1)} & \ldots & f_1^{(N)} \\ \vdots & \vdots & \ddots & \vdots \\ f_N^{(0)} & f_N^{(1)} & \ldots & f_N^{(N)} \\ 1 & \partial_x & \ldots & \partial_x^N \end{vmatrix} \tag{II.2}$$

provides a dressing operator[1] satisfying (I.2). In this formal determinant an expansion with respect to the last row is understood, in which all subdeterminants are collected on the left of the differential symbols. With this representation it is clear that the action of W on an arbitrary function g is given by

$$W(g) = \frac{\mathcal{W}(f_1, \ldots, f_N, g)}{\mathcal{W}(f_1, \ldots, f_N)} . \tag{II.3}$$

In particular, $W(f_1) = \ldots = W(f_N) = 0$, so that formula (II.2) provides the reconstruction of a normalized ordinary differential operator from its kernel.

We shall study the effect of the k-constraint on the entries of the Wronskian τ-function. We will make use of the following technical lemma:

Lemma: *The micro-differential inverse of the dressing operator (II.2) is given by*

$$W^{-1} = \sum_{i=1}^{N} f_i \, \partial_x^{-1} g_i$$

with

$$g_i = (-1)^{N-i} \frac{\mathcal{W}(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_N)}{\mathcal{W}(f_1, \ldots, f_N)} . \tag{II.4}$$

Proof: The functions g_i are the solutions of the linear system

$$\begin{pmatrix} f_1^{(0)} & \ldots & \ldots & f_N^{(0)} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ f_1^{(N-1)} & \ldots & \ldots & f_N^{(N-1)} \end{pmatrix} \begin{pmatrix} g_1 \\ \vdots \\ \vdots \\ g_N \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} .$$

[1] The finite micro-differential operator $W\partial_x^{-N} = 1 + w_1 \partial_x^{-1} + \cdots + w_N \partial_x^{-N}$ also satisfies (I.2) and is the dressing operator of Sato's theory.

From these equations and using the algebraic rule

$$f\partial_x^{-1} = \partial_x^{-1} f + \partial_x^{-2} f^{(1)} + \partial_x^{-3} f^{(2)} + \cdots$$

for the symbol ∂_x^{-1} one finds

$$\Big(W \sum_{i=1}^{N} f_i \, \partial_x^{-1} \, g_i \Big)_+ \; = \; \Big(W \sum_{k=0}^{\infty} \partial_x^{-k-1} \sum_{i=1}^{N} f_i^{(k)} \, g_i \Big)_+$$

$$= \; \Big(W \, \partial_x^{-N} \, \Big(1 \, + \, \sum_{k=N}^{\infty} \partial_x^{N-k-1} \sum_{i=1}^{N} f_i^{(k)} \, g_i \Big) \Big)_+ \; = \; 1 \; .$$

On the other hand,

$$\Big(W \sum_{i=1}^{N} f_i \, \partial_x^{-1} \, g_i \Big)_- \; = \; \sum_{i=1}^{N} \left(W(f_i) \right) \partial_x^{-1} \, g_i \; = \; 0 \; ,$$

whence $W \sum_i f_i \partial_x^{-1} g_i = 1$. Q.E.D.

The following theorem provides a simple criterion for the Wronskian entries $f_1, \ldots, f_N$ leading to k-constrained KP flows:

Theorem: *The KP solution $L = W\partial_x W^{-1}$ generated by the Wronskian τ-function $\mathcal{W}(f_1, \ldots, f_N) \neq 0$ satisfies the k-constraint (I.3) with some suitable functions $q_1, \ldots, q_M$ and $r_1, \ldots, r_M$ if and only if*

$$\mathcal{W}(f_1, \ldots, f_N, f_{i_1}^{(k)}, \ldots, f_{i_{M+1}}^{(k)}) = 0 \tag{II.5}$$

for all $(M+1)$–tupel of indices $1 \leq i_1 < \ldots < i_{M+1} \leq N$, which is equivalent to

$$\mathcal{W}\Big(\frac{\mathcal{W}(f_1, \ldots, f_N, f_{i_1}^{(k)})}{\mathcal{W}(f_1, \ldots, f_N)} \, , \ldots , \, \frac{\mathcal{W}(f_1, \ldots, f_N, f_{i_{M+1}}^{(k)})}{\mathcal{W}(f_1, \ldots, f_N)} \Big) \; = \; 0 \tag{II.6}$$

for all such indices.

Proof: The equivalence between (II.5) and (II.6) is provided by the Wronskian identity

$$\begin{aligned} &\mathcal{W}\Big(\frac{\mathcal{W}(f_1, \ldots, f_N, h_1)}{\mathcal{W}(f_1, \ldots, f_N)}, \ldots, \frac{\mathcal{W}(f_1, \ldots, f_N, h_{M+1})}{\mathcal{W}(f_1, \ldots, f_N)} \Big) \\ &= \; \frac{\mathcal{W}(f_1, \ldots, f_N, h_1, \ldots, h_{M+1})}{\mathcal{W}(f_1, \ldots, f_N)} \; , \end{aligned} \tag{II.7}$$

which holds for arbitrary functions $f_1, \ldots, f_N$ and $h_1, \ldots, h_{M+1}$. It is proven in the appendix.

Using the previous lemma on finds

$$(L^k)_- \;=\; (W \partial_x^k W^{-1})_- \;=\; \sum_{i=1}^{N} (W(f_i^{(k)}))\, \partial_x^{-1}\, g_i \tag{II.8}$$

with g_i defined by (II.4). Hence, the constraint (I.3) with $M = N$ is always satisfied for KP solutions generated by $N \times N$-Wronskians. It is claimed that the conditions (II.5)/(II.6) reduces the sum from N to M terms.

First, we show that (II.6) follows from (I.3). As already remarked in Section 1, for L satisfying (I.3) there exists an M-th order differential operator A such that AL^k is a differential operator. Application to $W(f_i) = 0$ yields

$$0 \;=\; AL^k(W(f_i)) \;=\; AW\partial_x^k(f_i) \;=\; A(W(f_i^{(k)}))\ .$$

Formula (II.3) leads to

$$W(f_i^{(k)}) \;=\; \frac{\mathcal{W}(f_1, \ldots, f_N, f_i^{(k)})}{\mathcal{W}(f_1, \ldots, f_N)} \;\in\; kernel(A)\ .$$

Since the kernel of A has dimension M, at most M of these functions can be linearly independent, whence (II.6) obtains. Conversely, if (II.6) holds, then at most M of the functions $W(f_i^{(k)})$ are linearly independent. Consequently, they can be expressed by M functions $q_1, \ldots, q_M$:

$$\frac{\mathcal{W}(f_1, \ldots, f_N, f_i^{(k)})}{\mathcal{W}(f_1, \ldots, f_N)} \;=\; \sum_{i=1}^{M} c_{ij}\, q_j\ , \quad i = 1, \ldots, N$$

with some constants c_{ij} and (II.8) reduces to

$$(L^k)_- \;=\; \sum_{i=1}^{N} \Big(\sum_{j=1}^{M} c_{ij}\, q_j\Big)\, \partial_x^{-1}\, g_i \;=\; \sum_{j=1}^{M} q_j\, \partial_x^{-1} \Big(\sum_{i=1}^{N} c_{ij}\, g_i\Big)\ . \tag{II.9}$$

Q.E.D.

III Example

We consider 2×2-Wronskians $\tau = \mathcal{W}(f_1, f_2)$ with

$$
\begin{aligned}
f_1 &= \exp(c_1 + \lambda_1 x + \lambda_1^2 y + \lambda_1^3 t) \; + \; \exp(d + \mu x + \mu^2 y + \mu^3 t) \; , \\
f_2 &= \exp(c_2 + \lambda_2 x + \lambda_2^2 y + \lambda_2^3 t) \; + \; \exp(c_3 + \lambda_3 x + \lambda_3^2 y + \lambda_3^3 t)
\end{aligned}
$$

with arbitrary constants c_i, λ_i, d, μ. These functions satisfy the linear equations (II.1) with $t_1 = x, t_2 = y$ and $t_3 = t$, whence $u(x, y, t) = \ln(\tau)_{xx}$ satisfies the KP equation

$$
4 u_{tx} \; = \; (u_{xxx} + 12 u u_x)_x + 3 u_{yy} \; . \tag{III.10}
$$

Using the theorem in the last section we will derive conditions on the parameters such that the k-constraint (I.3) with $M = 1$ is satisfied. The criterion (II.5) yields a single equation

$$
\begin{aligned}
0 \; &\stackrel{!}{=} \; \mathcal{W}(f_1, f_2, f_1^{(k)}, f_2^{(k)}) \\
&= \; (\mu^k - \lambda_1^k) \; (\lambda_2^k - \lambda_3^k) \; V(\lambda_1, \lambda_2, \lambda_3, \mu) \; e^{c_1 + c_2 + c_3 + d} \\
& \qquad \times \; e^{(\lambda_1 + \lambda_2 + \lambda_3 + \mu)\, x} \; e^{(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \mu^2)\, y} \; e^{(\lambda_1^3 + \lambda_2^3 + \lambda_3^3 + \mu^3)\, t}
\end{aligned}
$$

with the Vandermonde determinant

$$
V(\lambda_1, \lambda_2, \lambda_3, \mu) \; = \; \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 \\ 1 & \lambda_2 & \lambda_2^2 & \lambda_2^3 \\ 1 & \lambda_3 & \lambda_3^2 & \lambda_3^3 \\ 1 & \mu & \mu^2 & \mu^3 \end{vmatrix} \; .
$$

It may be satisfied by the choice $\mu = \lambda_2$. Putting $d = c_2$, the τ-function yields

$$
\tau = (\lambda_2 - \lambda_1)\, e^{\eta_1 + \eta_2} + (\lambda_3 - \lambda_2)\, e^{\eta_2 + \eta_3} + (\lambda_3 - \lambda_1)\, e^{\eta_1 + \eta_3} ,
$$

where

$$
\eta_i = c_i + \lambda_i \, x + \lambda_i^2 \, y + \lambda_i^3 \, t \; , \quad i = 1, 2, 3 \, .
$$

The constraint $(L^k)_- = q \partial_x^{-1} r$ is satisfied with

$$
q \; := \; (\lambda_3^k - \lambda_2^k) \; \frac{\mathcal{W}(f_1, f_2, f_1^{(k)})}{\mathcal{W}(f_1, f_2)} \; = \; (\lambda_1^k - \lambda_2^k) \; \frac{\mathcal{W}(f_1, f_2, f_2^{(k)})}{\mathcal{W}(f_1, f_2)}
$$

$$= \frac{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)(\lambda_1^k-\lambda_2^k)(\lambda_3^k-\lambda_2^k)}{(\lambda_2-\lambda_1)\,e^{-\eta_3}+(\lambda_3-\lambda_1)\,e^{-\eta_2}+(\lambda_3-\lambda_2)\,e^{-\eta_1}} ,$$

and, according to (II.9)/(II.4),

$$\begin{aligned} r \;:=&\; \Big(\frac{f_2}{(\lambda_2^k-\lambda_3^k)} + \frac{f_1}{(\lambda_1^k-\lambda_2^k)} \Big) \frac{1}{\mathcal{W}(f_1,f_2)} \\ =&\; \frac{e^{-\eta_1-\eta_2-\eta_3}}{(\lambda_1^k-\lambda_2^k)(\lambda_3^k-\lambda_2^k)} \, \frac{(\lambda_3^k-\lambda_2^k)\,e^{\eta_1}+(\lambda_3^k-\lambda_1^k)\,e^{\eta_2}+(\lambda_2^k-\lambda_1^k)\,e^{\eta_3}}{(\lambda_3-\lambda_2)\,e^{-\eta_1}+(\lambda_3-\lambda_1)\,e^{-\eta_2}+(\lambda_2-\lambda_1)\,e^{-\eta_3}} . \end{aligned}$$

These functions are regular for $\lambda_1 < \lambda_2 < \lambda_3$ and, for fixed y and t, bounded in x, if $\lambda_1 < 0$, $\lambda_2 = 0$, $\lambda_3 > 0$. The KP solution $u = \ln(\tau)_{xx}$ consists of three solitary waves located in (x,y,t)-space along the planes $\eta_1 \approx \eta_2$, $\eta_2 \approx \eta_3$ and $\eta_1 \approx \eta_3$, respectively. In particular, choosing $\lambda_1 = \lambda$, $\lambda_2 = 0$, $\lambda_3 = -\lambda$, $c_1 = c$, $c_2 = 0$, $c_3 = -c$ one finds

$$q \;=\; \frac{(-1)^k\,\lambda^{2(k+1)}}{1 \,+\, e^{-\lambda^2 y}\,\cosh(c+\lambda x+\lambda^3 t)}$$

and

$$r \;=\; \begin{cases} -\,\dfrac{1}{\lambda^{k+1}}\,\dfrac{e^{-\lambda^2 y}+\cosh(c+\lambda x+\lambda^3 t)}{e^{\lambda^2 y}+\cosh(c+\lambda x+\lambda^3 t)} & \text{for odd k ,} \\[2ex] -\,\dfrac{1}{\lambda^{k+1}}\,\dfrac{\sinh(c+\lambda x+\lambda^3 t)}{e^{\lambda^2 y}+\cosh(c+\lambda x+\lambda^3 t)} & \text{for even } k\ . \end{cases}$$

For $k = 1$ one has $L = \partial_x + q\partial_x^{-1} r$, whence q, r satisfy the first flows of the AKNS hierarchy

$$q_y \;=\; q_{xx} + 2q^2 r \ , \quad r_y \;=\; -\,r_{xx} - 2qr^2$$

and

$$q_t \;=\; q_{xxx} + 6qrq_x \ , \quad r_t \;=\; r_{xxx} + 6qrr_x \ ,$$

respectively. A solution of the KP equation (III.10) is given by $u = qr$. Plots of u and q in the (x,y)-plane at time $t = 0$ are given in Figures 1 and 2 for $\lambda_1 = -1, \lambda_2 = 0, \lambda_3 = 1$, $c_1 = c_2 = c_3 = 0$.

IV Concluding remarks

We have found necessary and sufficient conditions reducing a Wronskian solution of the general KP hierarchy to solutions of the k-constrained hierarchy. We have relied on properties of differential operators resulting from *finite* dressing operators. Of course, the same goal could have been achieved by formulating additional equations for general τ-functions which then might be specialized to τ-functions of Wronskian type. The constraints for general τ-functions, however, are technically more involved. For special cases some results exist: the simplest additional condition on the τ-function in the cases $k = 1$ and 2, respectively, is a trilinear form [12,13] and a quattrolinear form [13,14], respectively. For Wronskian τ-functions these general conditions reduce to the results of the theorem in Section 2.

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Appendix

We prove the Wronskian identity (II.7). With some given family of functions $f^{[n]}_{n+1}, f^{[n]}_{n+2}, \ldots$ one considers "iterated Darboux transformations" defined by the recursion

$$f_i^{[k+1]} := \frac{\mathcal{W}(f^{[k]}_{k+1}, f^{[k]}_i)}{f^{[k]}_{k+1}} , \quad i,k = n, n+1, \ldots .$$

Crum's classical result [15] is that after K such recursive steps the resulting functions have the Wronskian representation

$$f_i^{[n+K]} = \frac{\mathcal{W}(f^{[n]}_{n+1}, \ldots, f^{[n]}_{n+K}, f^{[n]}_i)}{\mathcal{W}(f^{[n]}_{n+1}, \ldots, f^{[n]}_{n+K})} . \tag{A.1}$$

Comparing the results for $(n,K) = (0, N+m)$ and $(n,K) = (N,m)$ one finds for $i = N+m+1$:

$$f^{[N+m]}_{N+m+1} = \frac{\mathcal{W}(f^{[0]}_1, \ldots, f^{[0]}_{N+m+1})}{\mathcal{W}(f^{[0]}_1, \ldots, f^{[0]}_{N+m})} = \frac{\mathcal{W}(f^{[N]}_{N+1}, \ldots, f^{[N]}_{N+m+1})}{\mathcal{W}(f^{[N]}_{N+1}, \ldots, f^{[N]}_{N+m})} . \tag{A.2}$$

Denoting $f^{[0]}_i = f_i$ for $i = 1, \ldots, N$ and $f^{[0]}_{N+i} = h_i$ for $i = 1, \ldots, m+1$ one has

$$f^{[N]}_{N+i} = \frac{\mathcal{W}(f_1, \ldots, f_N, h_i)}{\mathcal{W}(f_1, \ldots, f_N)} , \quad i = 1, \ldots, m+1$$

from (A.1) with $n = 0, K = N$ and (A.2) yields

$$f^{[N+m]}_{N+m+1} = \frac{\mathcal{W}(f_1, \ldots, f_N, h_1, \ldots, h_{m+1})}{\mathcal{W}(f_1, \ldots, f_N, h_1, \ldots, h_m)}$$

$$= \frac{\mathcal{W}\Big(\dfrac{\mathcal{W}(f_1,\ldots,f_N,h_1)}{\mathcal{W}(f_1,\ldots,f_N)}, \ldots, \dfrac{\mathcal{W}(f_1,\ldots,f_N,h_{m+1})}{\mathcal{W}(f_1,\ldots,f_N)} \Big)}{\mathcal{W}\Big(\dfrac{\mathcal{W}(f_1,\ldots,f_N,h_1)}{\mathcal{W}(f_1,\ldots,f_N)}, \ldots, \dfrac{\mathcal{W}(f_1,\ldots,f_N,h_m)}{\mathcal{W}(f_1,\ldots,f_N)} \Big)} \, .$$

Multiplication of these identities with $m = 0, \ldots, M$ yields (II.7). Q.E.D.

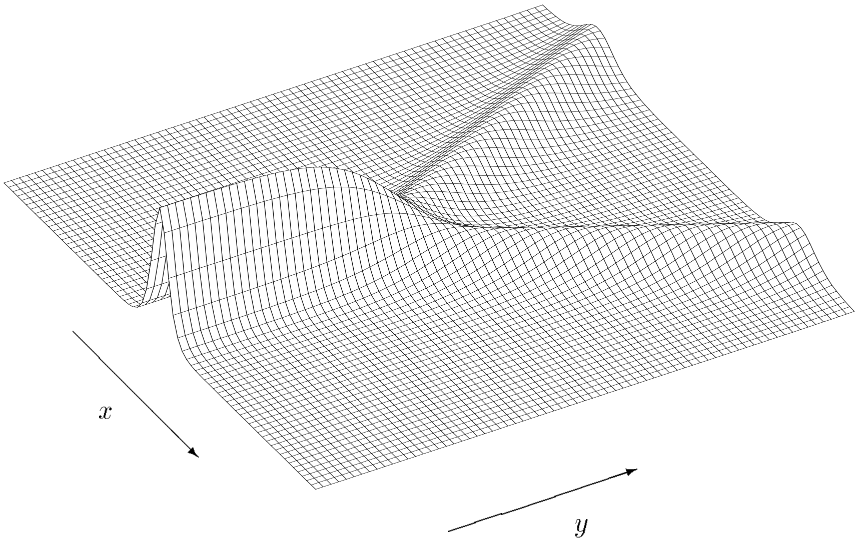

Figure 1: $u(x, y, t)$ for $t = 0$

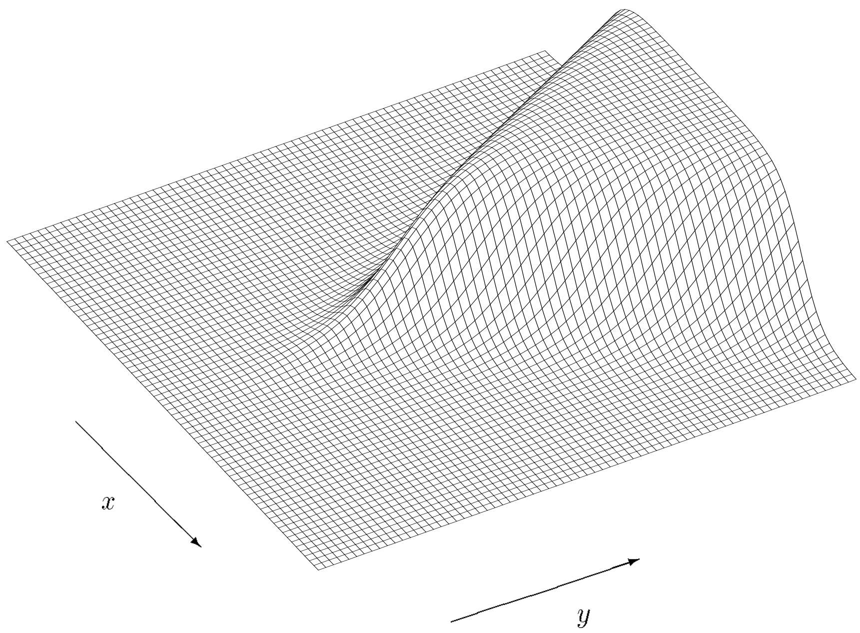

Figure 2: $q(x, y, t)$ for $t = 0$