Squared Eigenfunction Symmetries for Soliton Equations: Part II.

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Abstract

Nonlinear integrable evolution equations in $1+1$ dimensions arise from constraints of the $2+1$ dimensional hierarchies associated with the Kadomtsev-Petviashvili (KP) equation, the modified KP equation and the Dym equation, respectively. Links of Bäcklund type and of reciprocal type are known to exist between the 2+1 dimensional systems. The corresponding links between the constrained flows are discussed in a general framework. In particular, squared eigenfunction symmetries generated by solutions of the associated linear scattering problems are considered. The links between the soliton hierarchies are extended to these symmetries.

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1 Introduction

The present article is a sequel to Ref. [12] which will be referred to as "Part I" in the following. The notation and terminology of Part I is used.

Bäcklund and Reciprocal Transformations have been introduced in connection with boundary value problems, as well as initial value problems, arising in connection with models mainly in Gas Dynamics and Fluid Mechanics [13, 14]. For integrable systems, represented by a set of linear scattering equations, such transformations arise naturally as transformations of the underlying scattering operators. In Part I the Miura link between the KP hierarchy and the modified KP hierarchy as well as the reciprocal link between the modified KP hierarchy and the 2+1 dimensional Dym hierarchy were reviewed. These links were extended to the "squared eigenfunction symmetries", which are generated by products of eigenfunctions and adjoint eigenfunctions, i.e., solutions of the underlying linear scattering problems.

Building upon these 2+1 dimensional results, we now consider reductions to 1+1 dimensions. In recent years there has been considerable interest in constraints of the KP hierarchy induced by squared eigenfunctions [4, 5, 8, 9, 15, 16, 17]. We adopt the approach to these systems via Lax operators of pseudo-differential type which is a most convenient tool in 2+1 dimensions, also allowing a simple algebraic description of the constraints yielding 1+1 dimensional reductions.

In Section 2 the constraints of the three basic hierarchies (KP, modified KP and Dym) are formulated. The Miura links as well as the reciprocal links between the constrained flows are established. In Section 3 squared eigenfunction symmetries for the constrained flows are considered, the Miura relations and reciprocal links are extended to these symmetries.

The family of 1+1 dimensional soliton systems residing in the 2+1 hierarchies is large. The approach of this article provides a unified picture of the connections between equations in this family. This is demonstrated by various examples in which the general results are applied to specific Lax operators which correspond to specific integrable hierarchies.

The general results can be extended to even larger families of integrable equations, when multicomponent versions of the 2+1 hierarchies are used as a starting point. The restriction to scalar Lax operators in Parts I and II is mainly motivated by the existence of the interesting third class of equations (the Dym hierarchy) which is present only in the scalar case.

Proofs of all lemmata and theorems are given in the appendix.

2 Constrained hierarchies and their links

In this section we review the algebraic construction of the constrained hierarchies arising from symmetry reductions of the 2+1 dimensional systems defining the KP, the mKP and the Dym hierarchies.

We recall from Part I that the corresponding pseudo-differential Lax operators in 2+1 dimensions are

$$\begin{aligned}
k=0: \quad L_{\mathrm{KP}} \; &= \partial_x + u\,\partial_x^{-1} + u_2\,\partial_x^{-2} + u_3\,\partial_x^{-3} + \cdots \;, \\
k=1: \quad L_{\mathrm{mKP}} &= \partial_x + v + v_1\,\partial_x^{-1} + v_2\,\partial_x^{-2} + \cdots \;, \\
k=2: \quad L_{\mathrm{Dym}} &= w\,\partial_x + w_0 + w_1\,\partial_x^{-1} + w_2\,\partial_x^{-2} + \cdots \;,
\end{aligned} \tag{2.1}$$

where the leading fields u, v and w satisfy the various flows associated with the KP, the mKP and the Dym equations, respectively, when the Lax dynamics

$$L_{t_n} \;=\; [(L^n)_{\geq k}, L] \;, \quad k = 0, 1 \text{ or } 2 \;, \quad n \in \mathbb{N} \tag{2.2}$$

is imposed. Eigenfunctions ϕ and adjoint eigenfunctions ψ are defined as solutions of the associated linear evolution equations

$$\phi_{t_n} = [(L^n)_{\geq k}\phi] \;, \quad \psi_{t_n} = -[\partial_x^{-k}(L^n)^*_{\geq k}\partial_x^k\psi] \,. \tag{2.3}$$

Here, following the notation of Part I, the square brackets [*operator function*] indicate the action of a differential operator on a function. Important quantities associated with (adjoint) eigenfunctions are the potentials Ω and $\widehat{\Omega}$ defined by the compatible equations

$$\Omega(\psi^{(k)}, \phi)_x = \psi^{(k)}\phi \;, \quad \Omega(\psi^{(k)}, \phi)_{t_n} = \mathrm{res}(\partial_x^{-1}\psi^{(k)}(L^n)_{\geq k}\phi\,\partial_x^{-1}) \tag{2.4}$$

and

$$\widehat{\Omega}(\psi, \phi^{(k)})_x = \psi\phi^{(k)} \;, \quad \widehat{\Omega}(\psi, \phi^{(k)})_{t_n} = \mathrm{res}(\partial_x^{-1}\psi\partial_x^k(L^n)_{\geq k}\partial_x^{-k}\phi^{(k)}\,\partial_x^{-1}) \,, \tag{2.5}$$

respectively, where, in turn, $\psi^{(k)}$ denotes ψ, ψ_x or ψ_{xx} for $k = 0, 1$ or 2 (similarly for $\phi^{(k)}$). They are related by

$$\widehat{\Omega}(\psi, \phi^{(k)}) = \begin{cases} \Omega(\psi, \phi) & (k = 0) \ , \\ -\Omega(\psi_x, \phi) + \psi\phi & (k = 1) \ , \\ \Omega(\psi_{xx}, \phi) - \psi_x\phi + \psi\phi_x & (k = 2) \ . \end{cases}$$

As proven in Part I (lemma 2.2 and lemma 4.3), the compatibility conditions of these equations are the linear evolutions (2.3) of the (adjoint) eigenfunctions.

Reductions of the KP hierarchy induced by squared eigenfunctions [4, 5, 8, 9, 15, 16, 17] have been studied extensively in the literature. Such 1+1 dimensional systems arise by imposing a symmetry constraint involving solutions of the associated linear scattering problems (2.3). The corresponding Lax operators are obtained by restricting the pseudo-differential tail of the (infinite) pseudo-differential operators to some submanifold of a specific form. We formulate the constraints for the three classes of integrable hierarchies associated with the labels $k = 0, 1$ and 2, respectively:

Lemma 2.1: *Consider L given by (2.1), $k = 0, 1$ or 2. Assume*

$$(L^N)_{<k} = \sum_{j=1}^{m} q_j \partial_x^{-1} r_j \partial_x^k \tag{2.6}$$

for some $N \in \mathbb{N}$ and some functions $q_1, \ldots, q_m$, $r_1, \ldots, r_m$. Then the dynamics

$$L_{t_n} = [(L^n)_{\geq k}\, , L]\, , \quad q_{jt_n} = [(L^n)_{\geq k}\, q_j]\, , \quad r_{jt_n} = -[\partial_x^{-k} (L^n)^*_{\geq k} \partial_x^k\, r_j] \tag{2.7}$$

leaves the constraint (2.6) invariant.

The constrained dynamics is 1+1 dimensional, since (2.7) represents nonlinear evolution equations in the independent variables t_n and x for the $N-1$ fields parametrizing $(L^N)_{\geq k}$ and the fields $q_1, \ldots, q_m$, $r_1, \ldots, r_m$.

Example 2.2: We compute the equations of the first non-trivial flows in the hierarchies, obtained from $q_{t_2} = [(L^2)_{\geq k}\, q]$, $r_{t_2} = -[\partial_x^{-k}(L^2)^*_{\geq k}\partial_x^k\, r]$, associated with the following Lax operators ($N=1$). For later convenience we use tildes (bars) for objects associated with $k=1$ ($k=2$):

$$k=0: \;\; L = \partial_x + q\partial_x^{-1} r : \quad \begin{pmatrix} q \\ r \end{pmatrix}_{t_2} = \begin{pmatrix} q_{xx} + 2q^2 r \\ -r_{xx} - 2qr^2 \end{pmatrix} \tag{2.8}$$

$$k=1: \;\; \tilde{L} = \partial_x + \tilde{q}\partial_x^{-1}\tilde{r}\partial_x + \lambda : \quad \begin{pmatrix} \tilde{q} \\ \tilde{r} \end{pmatrix}_{t_2} = \begin{pmatrix} \tilde{q}_{xx} + 2\tilde{q}\tilde{r}\tilde{q}_x + \lambda\tilde{q}_x \\ -\tilde{r}_{xx} - 2\tilde{q}\tilde{r}\tilde{r}_x + \lambda\tilde{r}_x \end{pmatrix} \tag{2.9}$$

$$\tilde{L} = \partial_x + \tilde{q} + \partial_x^{-1}\tilde{r}\partial_x : \quad \begin{pmatrix} \tilde{q} \\ \tilde{r} \end{pmatrix}_{t_2} = \begin{pmatrix} \tilde{q}_{xx} + 2(\tilde{q}+\tilde{r})\,\tilde{q}_x \\ -\tilde{r}_{xx} + 2(\tilde{q}+\tilde{r})\,\tilde{r}_x \end{pmatrix} \tag{2.10}$$

$$k=2: \;\; \bar{L} = \bar{q}\partial_{\bar{x}}^{-1}\bar{r}\partial_{\bar{x}}^2 + (\tilde{\lambda}-\lambda)\,\bar{x}\,\partial_{\bar{x}} + \lambda :$$

$$\begin{pmatrix} \bar{q} \\ \bar{r} \end{pmatrix}_{\bar{t}_2} = \begin{pmatrix} (\bar{q}\bar{r} + (\tilde{\lambda}-\lambda)\,\bar{x}\,)^2\, \bar{q}_{\bar{x}\bar{x}} \\ -(\bar{q}\bar{r} + (\tilde{\lambda}-\lambda)\,\bar{x}\,)^2\, \bar{r}_{\bar{x}\bar{x}} \end{pmatrix} \tag{2.11}$$

$$\bar{L} = (\bar{q}+\bar{r}-\bar{x}^2)\,\partial_{\bar{x}} + \bar{x} - \bar{r}_{\bar{x}} + \partial_{\bar{x}}^{-1}\bar{r}_{\bar{x}\bar{x}} :$$

$$\begin{pmatrix} \bar{q} \\ \bar{r} \end{pmatrix}_{\bar{t}_2} = \begin{pmatrix} (\bar{q}+\bar{r}-\bar{x}^2)^2\, \bar{q}_{\bar{x}\bar{x}} \\ -(\bar{q}+\bar{r}-\bar{x}^2)^2\, \bar{r}_{\bar{x}\bar{x}} \end{pmatrix} . \tag{2.12}$$

The constant parameters $\lambda, \tilde{\lambda}$ are arbitrary. They will turn up naturally as spectral parameters in the following, when links between these equations and their hierarchies are considered. We note that all operators are of the form (2.6):

$$
\begin{aligned}
k=1: \quad & \tilde{L} = \partial_x + \tilde{q}\partial_x^{-1}\tilde{r}\partial_x + \lambda\,\partial_x^{-1}1\,\partial_x \ , \\
& \tilde{L} = \partial_x + \tilde{q}\partial_x^{-1}1\,\partial_x + 1\,\partial_x^{-1}\tilde{r}\partial_x \ , \\
k=2: \quad & \bar{L} = \bar{q}\partial_{\bar{x}}^{-1}\bar{r}\partial_{\bar{x}}^2 - \lambda\,\partial_{\bar{x}}^{-1}\bar{x}\partial_{\bar{x}}^2 + (\lambda+\tilde{\lambda})\,\partial_{\bar{x}}^{-1}1\,\partial_{\bar{x}}^2 \ , \\
& \bar{L} = 1\,\partial_{\bar{x}}^{-1}\bar{r}\partial_{\bar{x}}^2 - \bar{x}\partial_{\bar{x}}^{-1}\bar{x}\partial_{\bar{x}}^2 + \bar{q}\partial_{\bar{x}}^{-1}1\,\partial_{\bar{x}}^2 \ ,
\end{aligned}
$$

where the constant functions 1, λ and $\tilde{\lambda}$ are admissible (adjoint) eigenfunctions satisfying (2.3) for the cases $k=1,2$. For $k=2$ also $\bar{x}$ is a solution of (2.3). One encounters the AKNS system (2.8) for $k=0$. The cases $k=1$ lead to a derivative nonlinear Schroedinger system (2.9) [3] and the Kaup-Broer system (2.10) [10]. In example 2.7 we will derive links (2.8) $\to$ (2.10) $\to$ (2.12) between these equations and their higher flows, in example 2.9 links (2.8) $\to$ (2.9) $\to$ (2.11) will be established.

Remark 2.3: *The dynamics (2.7) of the constrained Lax operator (2.6) represents the compatibility condition of the linear equations*

$$\phi_{t_n} = [(L^n)_{\geq k}\phi]\,, \quad [(L^N)_{\geq k}\phi] + \textstyle\sum_j q_j\widehat{\Omega}(r_j,\phi^{(k)}) = \lambda\phi \tag{2.13}$$

and

$$\psi_{t_n} = -[\partial_x^{-k}(L^n)^*_{\geq k}\partial_x^k\psi]\,, \quad [\partial_x^{-k}(L^N)^*_{\geq k}\partial_x^k\psi] - (-1)^k \textstyle\sum_j \Omega(\psi^{(k)},q_j)r_j = \mu\psi\ ,$$

respectively. Here $\lambda,\mu\in\mathbb{C}$ are arbitrary spectral parameters and $\Omega(\,.\,,.)$, $\widehat{\Omega}(\,.\,,.)$ are the potentials defined by (2.4) and (2.5), respectively. The integro-

differential spectral problem in (2.13) may be reformulated as a matrix problem, e.g., for the scattering equation associated with $L = \partial_x + q\,\partial_x^{-1} r$, $k = 0$*:*

$$\left.\begin{aligned} \phi_x + q\,\widehat{\Omega}(r,\phi) &= 2\lambda\phi \\ \widehat{\Omega}(r,\phi)_x = r\phi \end{aligned}\right\} \quad \Longleftrightarrow \quad \begin{pmatrix} \widehat{\Omega} \\ \phi \end{pmatrix}_x = \begin{pmatrix} 0 & r \\ -q & 2\lambda \end{pmatrix} \begin{pmatrix} \widehat{\Omega} \\ \phi \end{pmatrix}$$

$$\Longleftrightarrow \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} \lambda & -q \\ r & -\lambda \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

with $\psi_1 = \exp(-\lambda x)\phi$, $\psi_2 = \exp(-\lambda x)\widehat{\Omega}$*. The latter is the standard scattering problem used to solve the AKNS hierarchy via Inverse Scattering techniques [1].*

Remark 2.4: *Considering the squared eigenfunction symmetry (see theorem 4.1 and lemma 4.3 in Part I)*

$$L_z = [M, L]\ , \quad M = \textstyle\sum_j q_j \partial_x^{-1} r_j \partial_x^k$$

and

$$\phi_z = \textstyle\sum_j q_j \widehat{\Omega}(r_j, \phi^{(k)})\ , \quad \psi_z = (-1)^k \sum_j \Omega(\psi^{(k)}, q_j) r_j$$

the constraint (2.6) can be regarded as a symmetry reduction $\partial/\partial t_N = -\partial/\partial z$ *of the 2+1 dimensional hierarchies discussed in Part I:*

$$\begin{gathered} L_z + L_{t_N} = [M + (L^N)_{\geq k}, L] = [M - (L^N)_{<k}, L] = 0\,, \\ \phi_z + \phi_{t_N} = \lambda\phi\,, \quad \psi_z + \psi_{t_N} = -\mu\psi\,. \end{gathered}$$

We now consider the links $k = 0 \to k = 1 \to k = 2$ of the three integrable soliton hierarchies as reviewed by theorem 3.2 in Part I (see also [7, 11]) together with the class of constraints (2.6):

Theorem 2.5: *Let $\widehat{\Omega}(\,.\,,.)$ be the potential defined by (2.5).*

a) (constrained KP $\to$ constrained mKP): Let $L = L_{\rm KP}$ satisfy the constraint $(L^N)_{<0} = \sum_{j=1}^{m} q_j \partial_x^{-1} r_j$. Then, for any function ϕ, the transformed operator $\tilde{L} = \phi^{-1} L \phi$ satisfies the constraint

$$(\tilde{L}^N)_{<1} = \sum_{j=1}^{m+1} \tilde{q}_j \partial_x^{-1} \tilde{r}_j \partial_x$$

with

$$\tilde{q}_j = \phi^{-1} q_j \,, \quad \tilde{r}_j = -\widehat{\Omega}(r_j, \phi) \,, \quad j = 1, \ldots, m$$

and

$$\tilde{q}_{m+1} = \phi^{-1} \Big([(L^N)_{\geq 0} \phi] + \sum_{j=1}^{m} q_j \widehat{\Omega}(r_j, \phi) \Big) \,, \quad \tilde{r}_{m+1} = 1 \,.$$

If L, q_j, r_j satisfy the constrained KP dynamics (2.7,$k=0$) and ϕ is an eigenfunction (i.e., $\phi_{t_n} = [(L^n)_{\geq 0} \phi]$), then, according to theorem 3.2.a) in Part I, the gauge transformed Lax operator $\tilde{L}$ satisfies the modified KP dynamics $\tilde{L}_{t_n} = [(\tilde{L}^n)_{\geq 1}, \tilde{L}]$. Further, $\tilde{q}_j$ and $\tilde{r}_j$ are new (adjoint) eigenfunctions:

$$\tilde{q}_{j t_n} = [(\tilde{L}^n)_{\geq 1} \, \tilde{q}_j] \,, \quad \tilde{r}_{j t_n} = [\partial_x^{-1} (\tilde{L}^n)^*_{\geq 1} \, \partial_x \, \tilde{r}_j] \,, \quad j = 1, \ldots, m+1 \,.$$

b) (constrained mKP $\to$ constrained Dym): Let $\tilde{L} = L_{\rm mKP}$ satisfy the constraint $(\tilde{L}^N)_{<1} = \sum_{j=1}^{m} \tilde{q}_j \partial_x^{-1} \tilde{r}_j \partial_x$. Then, for any transformation $x \to \bar{x} = \tilde{\phi}(x)$, the transformed operator $\bar{L} = \tilde{L}$ satisfies the constraint

$$(\bar{L}^N)_{\overline{<2}} = \sum_{j=1}^{m+1} \bar{q}_j \partial_{\bar{x}}^{-1} \bar{r}_j \partial_{\bar{x}}^2$$

with

$$\bar{q}_j = \tilde{q}_j \,, \quad \bar{r}_j = -\widehat{\Omega}(\tilde{r}_j, \tilde{\phi}_x) \,, \quad j = 1, \ldots, m$$

and

$$\bar{q}_{m+1} = [(\tilde{L}^N)_{\geq 1} \tilde{\phi}] + \sum_{j=1}^{m} \tilde{q}_j \widehat{\Omega}(\tilde{r}_j, \tilde{\phi}_x) \,, \quad \bar{r}_{m+1} = 1 \,.$$

We note $\bar{r}_{j\bar{x}} = -\tilde{r}_j$ *for* $j = 1, \ldots, m$. *If* $\tilde{L}$, $\tilde{q}_j$, $\tilde{r}_j$ *satisfy the modified KP dynamics (2.7,*$k = 1$*) and* $\tilde{\phi}$ *is an eigenfunction (i.e.,* $\tilde{\phi}_{t_n} = [(\tilde{L}^n)_{\geq 1} \tilde{\phi}]$*), then, according to theorem 3.2.b) in Part I, the transformed Lax operator* $\bar{L}$ *satisfies the Dym dynamics* $\bar{L}_{\bar{t}_n} = [(\bar{L}^n)_{\overline{\geq 2}}, \bar{L}]$ *with respect to the new independent variables* $\bar{x}$ *and* $\bar{t}_n = t_n$. *Further,* $\bar{q}_j$ *and* $\bar{r}_j$ *are new (adjoint) eigenfunctions:*

$$\bar{q}_{j\bar{t}_n} = [(\bar{L}^n)_{\overline{\geq 2}} \, \bar{q}_j] \,, \quad \bar{r}_{j\bar{t}_n} = -[\partial_{\bar{x}}^{-2} (\bar{L}^n)^{\overline{*}}_{\overline{\geq 2}} \, \partial_{\bar{x}}^2 \, \bar{r}_j] \,, \quad j = 1, \ldots, m+1 \,.$$

Here the projection $\overline{\geq 2}$ *and the transposition* $\overline{*}$ *are to be taken with respect to the transformed symbol* $\partial_{\bar{x}} = (\tilde{\phi}_x)^{-1} \, \partial_x$.

Remark 2.6: *One may choose one of the eigenfunctions in the pseudo-differential part of* L^N *to generate the transformation. E.g., with* $\phi = q_m$ *the transformation of the constrained KP hierarchy leads to*

$$(L^N)_{<0} = \sum_{j=1}^{m} q_j \partial_x^{-1} r_j \quad \rightarrow$$

$$(\tilde{L}^N)_{<1} = \sum_{j=1}^{m-1} \tilde{q}_j \partial_x^{-1} \tilde{r}_j \partial_x \; + \; 1 \, \partial_x^{-1} \tilde{r}_m \partial_x \; + \; \tilde{q}_{m+1} \partial_x^{-1} 1 \, \partial_x \,.$$

Similarly, the transformation $x \to \bar{x} = \tilde{q}_m$ yields

$$(\tilde{L}^N)_{<1} = \sum_{j=1}^{m} \tilde{q}_j \partial_x^{-1} \tilde{r}_j \partial_x \quad \to$$

$$(\bar{L}^N)_{\overline{<2}} = \sum_{j=1}^{m-1} \bar{q}_j \partial_{\bar{x}}^{-1} \bar{r}_j \partial_{\bar{x}}^2 \; + \; \bar{x}\, \partial_{\bar{x}}^{-1} \bar{r}_m\, \partial_{\bar{x}}^2 \; + \; \bar{q}_{m+1}\, \partial_{\bar{x}}^{-1} 1\, \partial_{\bar{x}}^2 \, .$$

Example 2.7: Starting with the KP reduction given by the Lax operator $L = \partial_x + q\partial_x^{-1} r$ and choosing the eigenfunction q for the transformation, the last remark yields

$$\tilde{L} = q^{-1} L q = \partial_x + 1\, \partial_x^{-1} \tilde{r} \partial_x + \tilde{q} \partial_x^{-1} 1\, \partial_x = \partial_x + \tilde{q} + \tilde{r} - \partial_x^{-1} \tilde{r}_x$$

with new (adjoint) eigenfunctions

$$\tilde{q} = q^{-1} q_x + \widehat{\Omega}(r, q)\,, \quad \tilde{r} = -\widehat{\Omega}(r, q)\,.$$

This yields the (spatial part of the) Miura transformation

$$\tilde{q} + \tilde{r} = q^{-1} q_x\,, \quad \tilde{r}_x = -rq$$

between the AKNS system (2.8) and the Kaup-Broer system (2.10) which are associated with L and $\tilde{L}$, respectively. Indeed, this connection may be verified directly by comparing the coefficients of

$$q^{-1} L q = \partial_x + q^{-1} q_x + \partial_x^{-1} r q = \tilde{L} = \partial_x + \tilde{q} + \tilde{r} - \partial_x^{-1} \tilde{r}_x\ .$$

Next, we consider the link to a reduction of the 2+1 dimensional Dym hierarchy. According to remark 2.6, using the eigenfunction $\tilde{\phi} = \tilde{q}$, the transformation $x \to \bar{x} = \tilde{q}$ maps $\tilde{L}$ to

$$\bar{L} = 1\,\partial_{\bar{x}}^{-1}\bar{r}\partial_{\bar{x}}^{2} - \bar{x}\partial_{\bar{x}}^{-1}\widehat{\Omega}(1,\tilde{q}_x)\partial_{\bar{x}}^{2} + \bar{q}\partial_{\bar{x}}^{-1}1\,\partial_{\bar{x}}^{2}$$

with

$$\bar{r} = -\widehat{\Omega}(\tilde{r},\tilde{q}_x)\,, \quad \bar{q} = \tilde{q}_x + \tilde{q}\,\widehat{\Omega}(1,\tilde{q}_x) + \widehat{\Omega}(\tilde{r},\tilde{q}_x) = \tilde{q}_x + \bar{x}\,\widehat{\Omega}(1,\tilde{q}_x) - \bar{r}\ .$$

We note $\widehat{\Omega}(1,\tilde{q}_x)_{\bar{x}} = 1$, whence $\widehat{\Omega}(\tilde{r},\tilde{q}_x) = \bar{x}+c$ with some arbitrary constant c. For $c = 0$ one finds the Lax operator $\bar{L}$ associated with (2.12), whence

$$\bar{r}_{\bar{x}} = -\bar{r}\,, \quad \bar{q} + \bar{r} - \bar{x}^2 = \tilde{q}_x = \frac{\partial \bar{x}}{\partial x}$$

yields a links between the Kaup-Broer system (2.10) and the system (2.12). This may be verified directly by comparing the coefficients of the Lax operators

$$\tilde{L} = \tilde{q}_x\partial_{\bar{x}} + \bar{x} + \tilde{r} - \partial_{\bar{x}}^{-1}\tilde{r}_{\bar{x}} = \bar{L} = (\bar{q} + \bar{r} - \bar{x}^2)\,\partial_{\bar{x}} + \bar{x} - \bar{r}_{\bar{x}} + \partial_{\bar{x}}^{-1}\bar{r}_{\bar{x}\bar{x}}.$$

Remark 2.8: *Alternatively, one may apply theorem 2.5 with an eigenfunction which satisfies a spectral equation (2.13). In this case the constraints are transformed as follows:*

$$(L^N)_{<0} = \sum_{j=1}^{m} q_j\partial_x^{-1}r_j \quad \to \quad (\tilde{L}^N)_{<1} = \sum_{j=1}^{m} \tilde{q}_j\partial_x^{-1}\tilde{r}_j\partial_x \ + \ \lambda$$

and

$$(\tilde{L}^N)_{<1} = \sum_{j=1}^{m} \tilde{q}_j\partial_x^{-1}\tilde{r}_j\partial_x \quad \to \quad (\bar{L}^N)_{\overline{<2}} = \sum_{j=1}^{m} \bar{q}_j\partial_{\bar{x}}^{-1}\bar{r}_j\partial_{\bar{x}}^{2} \ + \ \lambda\,\bar{x}\,\partial_{\bar{x}}\,.$$

Example 2.9: Starting with the KP reduction given by the Lax operator $L = \partial_x + q\partial_x^{-1}r$ and choosing an eigenfunction ϕ satisfying

$$\phi_x + q\,\widehat{\Omega}(r,\phi) = \lambda\,\phi \tag{2.14}$$

for the transformation, the last remark yields

$$\tilde{L} = \phi^{-1}L\phi = \partial_x + \tilde{q}\,\partial_x^{-1}\tilde{r}\partial_x + \lambda$$

with new (adjoint) eigenfunctions

$$\tilde{q} = \phi^{-1}q\,, \quad \tilde{r} = -\widehat{\Omega}(r,\phi)\,.$$

This link may be verified directly by comparing the coefficients of

$$\phi^{-1}L\phi = \partial_x + \phi^{-1}\phi_x + \phi^{-1}q\partial_x^{-1}r\phi = \tilde{L} = \partial_x + \tilde{q}\tilde{r} + \lambda - \tilde{q}\partial_x^{-1}\tilde{r}_x\,.$$

With $\phi = q/\tilde{q} = -\tilde{r}_x/r$ and (2.14) this yields the (spatial part of the) Miura transformation

$$qr + \tilde{q}\tilde{r}_x = 0\,, \quad q_x\tilde{q} - q\tilde{q}_x - q\tilde{r}\tilde{q}^2 = \lambda\,q\tilde{q}$$

between the AKNS system (2.8) and the derivative NLS system (2.9) which are associated with L and $\tilde{L}$, respectively.

Next, we consider the link of $\tilde{L}$ with a reduction of the 2+1 dimensional Dym hierarchy. According to the last remark the transformation $x \to \bar{x} = \tilde{\phi}$ with an eigenfunction $\tilde{\phi}$ satisfying

$$\tilde{\phi}_x + \tilde{q}\,\widehat{\Omega}(\tilde{r},\tilde{\phi}_x) + \lambda\,\tilde{\phi} = \tilde{\lambda}\,\tilde{\phi}$$

maps $\tilde{L} = \partial_x + \tilde{q}\partial_x^{-1}\tilde{r}\partial_x + \lambda\partial_x^{-1}1\,\partial_x$ to

$$\bar{L} = \bar{q}\partial_{\bar{x}}^{-1}\bar{r}\partial_{\bar{x}}^2 - \lambda\,\partial_{\bar{x}}^{-1}\widehat{\Omega}(1,\phi_x)\partial_{\bar{x}}^2 + \bar{\lambda}\,\bar{x}\,\partial_{\bar{x}} = \bar{q}\partial_{\bar{x}}^{-1}\bar{r}\partial_{\bar{x}}^2 + (\tilde{\lambda} - \lambda)\,\bar{x}\,\partial_{\bar{x}} + \lambda\,,$$

where $\bar{q} = \tilde{q}$, $\bar{r} = -\widehat{\Omega}(\tilde{r},\phi_x)$ and $\widehat{\Omega}(1,\phi_x) = \bar{x}$ is chosen (note that $\widehat{\Omega}(1,\phi_x)_{\bar{x}} = 1$). One obtains the link

$$\bar{q} = q\,, \quad \bar{r}_{\bar{x}} = -\tilde{r}\,, \quad \frac{\partial\bar{x}}{\partial x} = \bar{q}\bar{r} + (\tilde{\lambda} - \lambda)\,\bar{x}$$

between the derivative NLS system (2.9) and the system (2.11). Indeed, this is verified directly by comparing the coefficients of the Lax operators

$$\tilde{L} = \tilde{\phi}_x\partial_{\bar{x}} + \tilde{q}\,\partial_{\bar{x}}^{-1}\tilde{r}\partial_{\bar{x}} + \lambda = \bar{L} = (\bar{q}\bar{r} + (\tilde{\lambda} - \lambda)\,\bar{x})\,\partial_{\bar{x}} - \bar{q}\partial_{\bar{x}}^{-1}\bar{r}_{\bar{x}}\partial_{\bar{x}} + \lambda\,.$$

Example 2.10: Links among well known $1+1$ dimensional integrable hierarchies are retrieved easily from the general framework presented here. In particular, the Bäcklund and reciprocal transformations relating the KdV, mKdV and Dym hierarchies [2, 6] follow naturally (see also [7]). Consider the Lax operators

$$L^2 = \partial_x^2 + 2u\ , \quad \tilde{L}^2 = \partial_x^2 + 2v\partial_x\ , \quad \bar{L}^2 = w^2\partial_{\bar{x}}^2.$$

With $L = L_{\mathrm{KP}}$, $\tilde{L} = L_{\mathrm{mKP}}$, $\bar{L} = L_{\mathrm{Dym}}$ they may be regarded as the pseudo-differential Lax operators (2.1), subject to the constraint (2.6) with $N = 2, m = 0$. These operators generate the hierarchies of the KdV, the mKdV and the 1+1 dimensional Dym equation. In particular:

$$
\begin{array}{llll}
(L^2)_{t_3} = [(L^3)_{\geq 0}, L^2] & \Longleftrightarrow & 4u_{t_3} = u_{xxx} + 12\, u u_x & \text{(KdV)},\\
(\tilde{L}^2)_{t_3} = [(\tilde{L}^3)_{\geq 1}, \tilde{L}^2] & \Longleftrightarrow & 4v_{t_3} = v_{xxx} - 6\, v^2 v_x & \text{(mKdV)},\\
(\bar{L}^2)_{t_3} = [(\bar{L}^3)_{\geq 2}, \bar{L}^2] & \Longleftrightarrow & 4w_{\bar{t}_3} = w^3 w_{\bar{x}\bar{x}\bar{x}} & \text{(Dym)}.
\end{array}
$$

Invoking theorem 2.5.a) with an eigenfunction ϕ satisfying the spectral equation $[L^2\phi] = \phi_{xx} + 2u\phi = 0$ on computes $\tilde{L}^2 = \phi^{-1} L^2 \phi = \partial_x^2 + 2\,\phi^{-1}\phi_x\,\partial_x$, i.e., $v = \phi^{-1}\phi_x$. Elimination of ϕ by means of the spectral equation yields the standard Miura link

$$
v_x + v^2 \;=\; \frac{\phi_{xx}}{\phi} \;=\; -\,2\,u
$$

between the KdV solution u and the mKdV solution v. With an eigenfunction $\tilde{\phi}$ satisfying the spectral equation $[\tilde{L}^2\,\tilde{\phi}] = \tilde{\phi}_{xx} + 2\,v\,\tilde{\phi}_x = 0$ the reciprocal transformation $\bar{x} = \tilde{\phi}$ of theorem 2.5.b) yields $\bar{L}^2 = \tilde{L}^2 = \tilde{\phi}_x^2 \partial_{\bar{x}}^2$, whence $w = \tilde{\phi}_x$ satisfies the Dym equation. Elimination of $\tilde{\phi}$ by means of the spectral equation yields the reciprocal link between the mKdV solution v and the Dym solution w characterized by

$$
\frac{\partial\bar{x}}{\partial x} \;=\; \tilde{\phi}_x \;=\; w\;, \quad w_{\bar{x}} \;=\; \frac{w_x}{w} \;=\; \frac{-2\,v\,\tilde{\phi}_x}{w} \;=\; -\,2\,v\;.
$$

3 Squared eigenfunction symmetries and their links

Here we consider the squared eigenfunction symmetries associated with the constrained hierarchies of the previous section and discuss the additional conditions arising from the constraint (2.6).

We assume that (adjoint) eigenfunctions $\phi_i,\ \psi_i,\ i = 1,\ldots,m_1$ satisfying (2.3) are given. They generate the squared eigenfunction flow as defined in theorem 4.1 of Part I, which is compatible with the Lax dynamics (2.2). One has the following result concerning the invariance of the constraint (2.6):

Theorem 3.1: *The constraint (2.6) is invariant under the squared eigenfunction flow*

$$L_z = \Big[\ \sum_{i=1}^{m_1} \phi_i \partial_x^{-1} \psi_i \partial_x^k\,,\ L\ \Big]\,,$$

$$q_{jz} = \sum_{i=1}^{m_1} \phi_i \widehat{\Omega}(\psi_i, q_j^{(k)})\,, \quad r_{jz} = (-1)^k \sum_{i=1}^{m_1} \Omega(r_j^{(k)}, \phi_i)\psi_i\,,$$

if $\phi_1,\ldots,\phi_{m_1}$ *and* $\psi_1,\ldots,\psi_{m_1}$ *satisfy*

$$[(L^N)_{\geq k}\,\phi_i] + \sum_{j=1}^{m} q_j \widehat{\Omega}(r_j, \phi_i^{(k)}) = \lambda_i \phi_i\,,$$

$$[\partial_x^{-k}(L^N)^*_{\geq k}\,\partial_x^k\,\psi_i] - (-1)^k \sum_{j=1}^{m} \Omega(\psi_i^{(k)}, q_j) r_j = \lambda_i \psi_i$$

with arbitrary spectral parameters $\lambda_i \in \mathbb{C}$.

We finally consider the standard constraints, where $(L^N)_{<k} = 0$, i.e., the Lax operator L^N is purely differential. According to theorem 3.2 of Part I one may use an eigenfunction ϕ, say, to trigger a transformation $k = 0 \to k = 1$ or $k = 1 \to k = 2$ between the different Lax hierarchies. According to theorem 4.6 in Part I the transformation is capable of linking the squared eigenfunction symmetries, provided the triggering eigenfunction ϕ satisfies

a certain z-evolution. Specifically, for a squared eigenfunction symmetry generated by pairs of (adjoint) eigenfunctions ϕ_i and ψ_i, one has to impose the flow

$$\phi_z = \sum_i \phi_i \widehat{\Omega}(\psi_i, \phi^{(k)}) \tag{3.15}$$

on the triggering function. Following remark 2.8 it is natural to consider solutions of a spectral problem $[L^N\phi] = \lambda\phi$ in order to avoid the appearance of pseudo-differential terms introduced by the transformation triggered by ϕ. One must guarantee that the flow (3.15) is consistent with the spectral condition $[L^N\phi] = \lambda\phi$ needed to preserve the differential character of the operator. It turns out that this is granted via a particular choice of the arbitrary constant in the potentials $\widehat{\Omega}(\psi_i, \phi^{(k)})$. Indeed, due to the spectral equations the squared eigenfunction potentials can be represented in the following way:

Lemma 3.2: *Let L satisfy the constraint $(L^N)_{<k} = 0$ for $k = 0, 1$ or 2. Let L evolve according to (2.2), let ϕ, ψ be (adjoint) eigenfunctions satisfying the dynamics (2.3) as well as the spectral equations*[1] $[L^N\phi] = \lambda\,\phi$ *and $[\partial_x^{-k}(L^N)^*\partial_x^k\psi] = \mu\,\psi$ with spectral parameters $\lambda, \mu \in \mathbb{C}$. Then, for $\mu \neq \lambda$, the potentials defined by (2.4) and (2.5) may be chosen as*

$$\Omega(\psi^{(k)}, \phi) = \frac{\mathrm{res}(\partial_x^{-1}\psi^{(k)}L^N\phi\,\partial_x^{-1})}{\lambda - \mu}$$

[1]No boundary conditions are assumed, i.e., ϕ and ψ need not be L_2-functions.

and

$$\widehat{\Omega}(\psi, \phi^{(k)}) = \frac{\mathrm{res}(\partial_x^{-1} \psi \partial_x^k L^N \partial_x^{-k} \phi^{(k)} \, \partial_x^{-1})}{\lambda - \mu}$$

respectively.

With these choices of the potentials one obtains:

Theorem 3.3: *Let L satisfy the constraint $(L^N)_{<k} = 0$ for $k = 0, 1$ or 2. Let $\phi_1, \ldots, \phi_m$ and $\psi_1, \ldots, \psi_m$ be (adjoint) eigenfunctions satisfying the spectral equations $[L^N \phi_i] = \lambda_i \phi_i$ and $[\partial_x^{-k} (L^N)^* \partial_x^k \psi_i] = \lambda_i \psi_i$, respectively. Consider the squared eigenfunction flow $L_z = [\sum_i \phi_i \partial_x^{-1} \psi_i \partial_x^k, L]$. Then the equations*

$$[L^N \phi] = \lambda \phi \quad \text{and} \quad \phi_z = \textstyle\sum_i \phi_i \widehat{\Omega}(\psi_i, \phi^{(k)})$$

are compatible, provided that the potentials are chosen according to

$$(\lambda - \lambda_i) \, \widehat{\Omega}(\psi_i, \phi^{(k)}) = \mathrm{res}(\partial_x^{-1} \psi_i \partial_x^k L^N \partial_x^{-k} \phi^{(k)} \partial_x^{-1}) \,, \tag{3.16}$$

or, equivalently,

$$(\lambda - \lambda_i) \, \Omega(\psi_i^{(k)}, \phi) = \mathrm{res}(\partial_x^{-1} \psi_i^{(k)} L^N \phi \partial_x^{-1}) \,.$$

According to lemma 3.2 the requirement (3.16) is indeed consistent with the defining equations (2.5) for $\lambda \neq \lambda_i$.

Example 3.4: Under the constraint $(L^2)_{<0} = 0$ for $L = L_{\mathrm{KP}}$, i.e., $L^2 = \partial_x^2 + 2u$, the KP hierarchy reduces to the KdV equation $4u_{t_3} = u_{xxx} + 12uu_x$ and its higher flows. Let $\phi_1, \ldots, \phi_m$ and $\psi_1, \ldots, \psi_m$ solve

$$[L^2\phi_i] = \phi_{ixx} + 2u\,\phi_i = \lambda_i\phi_i\,, \quad [L^2\psi_i] = \psi_{ixx} + 2u\,\psi_i = \lambda_i\psi_i\,.$$

They generate the squared eigenfunction flow $(L^2)_z = [\sum_i \phi_i\partial_x^{-1}\psi_i, L^2]$ which yields the system

$$u_z = -\textstyle\sum_i(\phi_i\psi_i)_x\,, \quad \phi_{ixx} + u\phi_i = \lambda_i\phi_i\,, \quad \psi_{ixx} + u\psi_i = \lambda_i\psi_i$$

characterizing this symmetry of the KdV hierarchy[2]. We apply theorem 4.6.a) of Part I with an eigenfunction ϕ satisfying

$$[L^2\phi] = \phi_{xx} + u\phi = \lambda\phi\;, \quad \phi_z = \sum_i \phi_i\widehat{\Omega}(\psi_i, \phi)\;,$$

where according to theorem 3.3 one has to choose

$$\widehat{\Omega}(\psi_i, \phi) = \frac{\mathrm{res}(\partial_x^{-1}\psi_i L^2\phi\partial_x^{-1})}{\lambda - \lambda_i} = \frac{\psi_i\phi_x - \psi_{ix}\phi}{\lambda - \lambda_i}.$$

for $\lambda \neq \lambda_i$. If $\lambda = \lambda_i$ for some index i, then $\phi = const\ \psi_i$ needs to be chosen[3] to grant the compatibility of the z-flow for ϕ with $[L^2\phi] = \lambda\phi$. The map

[2]The recursion operator $R(u) : \sigma \to \sigma_{xx} + 8u\sigma + 4u_x \int_{-\infty}^{x} \sigma$ generates the hierarchy of higher KdV flows. We note $R(u)u_z = 4u_x(\sum\phi_i\psi_i)_{|x=-\infty} - 4\sum\lambda_i\phi_i\psi_i$. For $\lambda_i = 0$ one has $R(u)u_z = const\ u_x$, whence this special squared eigenfunction symmetry may be regarded as the first "negative flow" of the KdV hierarchy.

[3]Note that eigenfunctions and adjoint eigenfunctions may be identified, since L^2 is symmetric. In this case the requirement (3.16) is satisfied automatically (the residue vanishes due to the symmetry of L^2), whence no condition on the arbitrary constant in $\widehat{\Omega}(\psi_i, \phi)$ arises.

$L \to \tilde{L} = \phi^{-1} L \phi$ yields the reduction

$$\tilde{L}^2 = \partial_x^2 + 2\phi^{-1}\phi_x \partial_x + \lambda$$

of the modified KP hierarchy, whence $v = \phi^{-1}\phi_x$ satisfies the modified KdV equation $4v_{t_3} = v_{xxx} - 6v^2 v_x + 6\lambda v_x$ and its hierarchy. According to theorem 4.6.a) in Part I one obtains the modified z-flow $\tilde{L}_z = [\sum_j \tilde{\phi}_i \partial_x^{-1} \tilde{\psi}_i \partial_x, \tilde{L}]$ with $\tilde{\phi}_i = \phi^{-1}\phi_i$, $\tilde{\psi}_i = -\widehat{\Omega}(\psi_i, \phi)$, which again satisfy the spectral equations $[\tilde{L}\tilde{\phi}_i] = \lambda_i \tilde{\phi}_i$ and $[\partial_x^{-1}(\tilde{L}^2)^* \partial_x \tilde{\psi}_i] = \lambda_i \tilde{\psi}_i$. This Lax equation yields the system

$$v_z = -\sum_i (\tilde{\phi}_i \tilde{\psi}_i)_x \,, \quad \tilde{\phi}_{ixx} + 2v\tilde{\phi}_{ix} = (\lambda_i - \lambda)\phi_i \,, \quad \tilde{\psi}_{ixx} - 2v\tilde{\psi}_{ix} = (\lambda_i - \lambda)\psi_i \,,$$

representing the squared eigenfunction symmetry of the modified KdV hierarchy[4]. For the special case given by only $m = 2$ pairs, where we choose $\tilde{\phi}_1 = -1$, $\tilde{\psi}_2 = 1$, $\lambda_1 = \lambda_2 = \lambda$, one obtains

$$v_z = \tilde{\psi}_{1x} - \tilde{\phi}_{2x} \,, \quad \tilde{\psi}_{1xx} = 2v\tilde{\psi}_{1x} \,, \quad \tilde{\phi}_{2xx} = -2v\tilde{\phi}_{2x} \,, \tag{3.17}$$

i.e.,

$$V_{zx} = a - b \,, \quad a_x = 2V_x a \,, \quad b_x = -2V_x b \,,$$

where $V = \ln(\phi)$ (i.e., $v = V_x$), $a = \tilde{\psi}_{1x}$ and $b = \tilde{\phi}_{2x}$. Integration yields $a = f(z)\exp(2V)$, $b = g(z)\exp(-2V)$ with arbitrary functions $f(z)$ and $g(z)$, i.e.,

$$V_{zx} = f(z)\exp(2V) - g(z)\exp(2V) \,.$$

[4]With $v = \ln(\phi)_x$ the z-evolution is nothing but $(\phi^{-1}(\phi_z - \sum \phi_i \widehat{\Omega}(\psi_i, \phi)))_x = 0$.

For $f(z) = g(z)$ (characterized by $ba = \tilde{\phi}_{2x}\tilde{\psi}_{1x} = 1$) one obtains the sinh-Gordon equation $V_{zx} = \sinh(2V)$ after an appropriate rescaling of z, which thus has been identified as the special squared eigenfunction symmetry (3.17) of the modified KdV hierarchy, which is subject to the constraint $\tilde{\phi}_{2x}\tilde{\psi}_{1x} = 1$. For $f(z) = 0$ or $g(z) = 0$ one obtains the Liouville equations $V_{zx} = \exp(\pm 2V)$.

A link to the squared eigenfunction symmetry of the constrained Dym hierarchy is obtained by theorem 4.6.b) in Part I. We choose an eigenfunction $\tilde{\phi}$ satisfying $[\tilde{L}^2\tilde{\phi}] = \tilde{\lambda}\phi$, i.e.,

$$\tilde{\phi}_{xx} + 2v\tilde{\phi}_x = (\tilde{\lambda} - \lambda)\,\tilde{\phi}$$

and $\tilde{\phi}_z = \sum_i \tilde{\phi}_i \widehat{\Omega}(\tilde{\psi}_i, \tilde{\phi}_x)$ with

$$\widehat{\Omega}(\tilde{\psi}_i, \tilde{\phi}_x) = \frac{\mathrm{res}(\partial_x^{-1}\tilde{\psi}_i\partial_x\tilde{L}^2\partial_x^{-1}\tilde{\phi}_x\partial_x^{-1})}{\tilde{\lambda} - \lambda_i} = \frac{(\tilde{\lambda} - \lambda)\,\tilde{\psi}_i\tilde{\phi} - \tilde{\psi}_{ix}\tilde{\phi}_x}{\tilde{\lambda} - \lambda_i}\,,$$

whence $\tilde{\phi}_z$ is compatible with $[\tilde{L}^2\tilde{\phi}] = \tilde{\lambda}\tilde{\phi}$ (theorem 3.3). The map $\tilde{L} \to \bar{L} = \tilde{L}$ induced by $\bar{x} = \tilde{\phi}$ yields the reduction

$$\bar{L}^2 = w^2\partial_{\bar{x}}^2 + (\tilde{\lambda} - \lambda)\,\bar{x}\,\partial_{\bar{x}} + \lambda\,, \quad w = \tilde{\phi}_x$$

of the 2+1 dimensional Dym hierarchy, whence w satisfies the 1+1 dimensional Dym hierarchy

$$w_{\bar{t}_2} = (\tilde{\lambda} - \lambda)\,(w - \bar{x}\,w_{\bar{x}})\,, \quad 4\,w_{\bar{t}_3} = w^3 w_{\bar{x}\bar{x}\bar{x}} + 3\,\bar{x}\,(\tilde{\lambda} - \lambda)^2\left(1 - \frac{\bar{x}\,w_{\bar{x}}}{w}\right) \quad (3.18)$$

and the corresponding higher isospectral flows. Applying theorem 4.6.b) of Part I the squared eigenfunction symmetry is obtained: $\bar{L}_{\bar{z}} =$

$[\sum_i \bar{\phi}_i \partial_{\bar{x}}^{-1} \bar{\psi}_i \partial_{\bar{x}}^2 \,,\, \bar{L}\,]$, wherein $\bar{z} = z$, $\bar{\phi}_i = \tilde{\phi}_i$ and

$$\bar{\psi}_i = -\widehat{\Omega}(\tilde{\psi}_i, \tilde{\phi}_x) = \frac{\tilde{\psi}_{ix} w - (\tilde{\lambda} - \lambda)\, \bar{x}\, \tilde{\psi}_i}{\tilde{\lambda} - \lambda_i}\,, \tag{3.19}$$

which satisfy $[\bar{L}^2 \bar{\phi}_i] = \lambda_i \bar{\phi}_i$ and $[\partial_{\bar{x}}^{-2} (\bar{L}^2)^* \partial_{\bar{x}}^2 \bar{\psi}_i] = \lambda_i \bar{\psi}_i$. This symmetry of (3.18) is characterized by the system

$$w_{\bar{z}} = \textstyle\sum_i (\, \bar{\phi}_i \bar{\psi}_i w_{\bar{x}} - w(\bar{\phi}_i \bar{\psi}_i)_{\bar{x}}\,)\,,$$

$$w^2 \bar{\phi}_{i\bar{x}\bar{x}} + (\tilde{\lambda} - \lambda) \bar{x}\, \bar{\phi}_{i\bar{x}} = (\lambda_i - \lambda)\, \bar{\phi}_i\,, \quad w^2 \bar{\psi}_{i\bar{x}\bar{x}} + (\lambda - \tilde{\lambda}) \bar{x}\, \bar{\psi}_{i\bar{x}} = (\lambda_i - \tilde{\lambda})\, \bar{\psi}_i\,.$$

We finally discuss the case arising from the sinh-Gordon equation, which is the squared eigenfunction symmetry of the modified KdV hierarchy generated by two pairs $(\tilde{\phi}_1, \tilde{\psi}_1)$ and $(\tilde{\phi}_2, \tilde{\psi}_2)$ with $\tilde{\phi}_1 = -1$, $\tilde{\psi}_2 = 1$, $\lambda_1 = \lambda_2 = \lambda$ and $\tilde{\phi}_{2x} \tilde{\psi}_{1x} = 1$. This leads to $\bar{\phi}_1 = -1$ and $\bar{\psi}_2 = -\bar{x}$ from (3.19), whence the corresponding equation compatible with the Dym hierarchy is

$$w_{\bar{z}} = w(\bar{\psi}_1 + \bar{x} \bar{\phi}_2)_{\bar{x}} - w_{\bar{x}} (\bar{\psi}_1 + \bar{x} \bar{\phi}_2)\,,$$

$$w^2 \bar{\phi}_{2\bar{x}\bar{x}} = (\lambda - \tilde{\lambda})\, \bar{x}\, \bar{\phi}_{2\bar{x}}\,, \quad w^2 \bar{\psi}_{1\bar{x}\bar{x}} = (\tilde{\lambda} - \lambda)\, (\bar{x}\, \bar{\psi}_{1\bar{x}} - \bar{\psi}_1)\,,$$

constrained by

$$-w^2\, \bar{\phi}_{2\bar{x}}\, \bar{\psi}_{2\bar{x}\bar{x}} = (\lambda - \tilde{\lambda})\, \bar{\phi}_{2\bar{x}}\, (\bar{x}\, \bar{\psi}_{1\bar{x}} - \bar{\psi}_1) = 1$$

(note $w \bar{\phi}_{2\bar{x}} = \tilde{\phi}_{2x}$ and $w \bar{\psi}_{1\bar{x}\bar{x}} = -\tilde{\psi}_{1x}$). The transformation from (3.17) is given by

$$w = \frac{\partial \bar{x}}{\partial x}\,, \quad \bar{\phi}_2 = \tilde{\phi}_2\,, \quad \bar{\psi}_1 = \frac{\tilde{\psi}_{1x} w}{\tilde{\lambda} - \lambda} - \bar{x}\, \tilde{\psi}_1\,, \quad w(w_{\bar{x}} + 2v) = (\tilde{\lambda} - \lambda)\, \bar{x}\,,$$

where the last equation is the spectral problem $\tilde{\phi}_{xx} + 2v\tilde{\phi}_x = (\tilde{\lambda} - \lambda)\tilde{\phi}$ for $w = \tilde{\phi}_x$.

4 Conclusions

We have studied the three families of 1+1 dimensional integrable soliton equations which arise as constrained flows of the KP hierarchy, the modified KP hierarchy and the Dym hierarchy, respectively. The Miura links between constrained KP ($k = 0$) and constrained modified KP ($k = 1$) equations are given by a simple gauge transformation of the associated Lax operators involving an eigenfunction. The reciprocal link between the systems of modified KP class ($k = 1$) and Dym class ($k = 2$) consists of a mere reparametrization of the independent variable x by means of an eigenfunction. These results extend to the squared eigenfunction symmetries augmenting the soliton hierarchies. By specifying a Lax operator and applying the transformations described in the general theorems such links may be derived for large families of soliton hierarchies. Thus, the approach adopted here yields a unified picture of transformations of constrained KP flows to related (modified) integrable systems.

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Appendix

In the following computations we refer to the identities (A.1), ... ,(A.9) as given in the appendix of Part I.

Proof of lemma 2.1: With

$$
\begin{aligned}
&[\,(L^n)_{\geq k}, L^N]_{<k} \;=\; [\,(L^n)_{\geq k}, (L^N)_{<k}]_{<k} \;=\; [\,(L^n)_{\geq k}, \textstyle\sum_j q_j\partial_x^{-1} r_j \partial_x^k]_{<k} \\
&= \textstyle\sum_j \Big((L^n)_{\geq k} q_j \partial_x^{-1}\Big)_{<0} r_j\partial_x^k \;-\; \textstyle\sum_j q_j\Big(\partial_x^{-1} r_j \partial_x^k (L^n)_{\geq k}\Big)_{<k} \\
&\overset{\text{(A.8)}}{=} \textstyle\sum_j [(L^n)_{\geq k} q_j]\,\partial_x^{-1} r_j\partial_x^k \;-\; \textstyle\sum_j q_j\Big(\partial_x^{-1} r_j \partial_x^k (L^n)_{\geq k}\partial_x^{-k}\Big)_{<0}\partial_x^k \\
&\overset{\text{(A.9)}}{=} \textstyle\sum_j [(L^n)_{\geq k} q_j]\,\partial_x^{-1} r_j\partial_x^k \;-\; \textstyle\sum_j q_j\partial_x^{-1}[\partial_x^{-k}(L^n)^*_{\geq k}\partial_x^k\, r_j]\,\partial_x^k \\
&= \tfrac{d}{dt_n}\textstyle\sum_j \; q_j\partial_x^{-1} r_j\partial_x^k
\end{aligned}
$$

one finds

$$
\tfrac{d}{dt_n}\;\Big((L^N)_{<k} - \textstyle\sum_j q_j\partial_x^{-1} r_j\partial_x^k\Big) = [\,(L^n)_{\geq k}, L^N]_{<k} - \tfrac{d}{dt_n}\textstyle\sum_j q_j\partial_x^{-1} r_j\partial_x^k = 0\,,
$$

whence the constraint (2.6) is invariant under the dynamics (2.7).

$\square$

Proof of theorem 2.5: a) One computes the new constraint

$$
\begin{aligned}
(\tilde{L}^N)_{<1} \;&=\; \phi^{-1}(L^N\phi)_{<1} \;=\; \phi^{-1}(L^N\phi)_0 + \phi^{-1}(L^N)_{<0}\phi \\
&=\; \phi^{-1}[(L^N)_{\geq 0}\phi] + \phi^{-1}\sum_{j=1}^{m} q_j\partial_x^{-1} r_j\phi \\
&=\; \phi^{-1}[(L^N)_{\geq 0}\phi] + \sum_{j=1}^{m}\phi^{-1} q_j\,(\,\widehat{\Omega}(r_j,\phi) - \partial_x^{-1}\widehat{\Omega}(r_j,\phi)\partial_x\,) \\
&=\; \phi^{-1}\Big([(L^N)_{\geq 0}\phi] + \sum_{j=1}^{m} q_j\widehat{\Omega}(r_j,\phi)\Big) + \sum_{j=1}^{m}\tilde{q}_j\partial_x^{-1}\tilde{r}_j\partial_x\,.
\end{aligned}
$$

The dynamics of $\tilde{q}_j$, $\tilde{r}_j$ with $j = 1,\ldots,m$ is derived by a simple computation. The modified KP dynamics of $\tilde{L}^N$ grants the eigenfunction dynamics for $\tilde{q}_{m+1}$ as well.

b) One computes the new constraint

$$
\begin{aligned}
(\bar{L}^N)_{\overline{<2}} &= (\bar{L}^N)_{\overline{<1}} + [(\bar{L}^N)_{\overline{\geq 1}}\,\bar{x}]\,\partial_{\bar{x}} \;=\; (\bar{L}^N)_{<1} + [(\tilde{L}^N)_{\geq 1}\,\tilde{\phi}]\,\partial_{\bar{x}} \\
&= \sum_{j=1}^{m} \tilde{q}_j \partial_x^{-1} \tilde{r}_j \partial_x + [(\tilde{L}^N)_{\geq 1}\,\tilde{\phi}]\,\partial_{\bar{x}} \\
&= -\sum_{j=1}^{m} \bar{q}_j \partial_{\bar{x}}^{-1} \bar{r}_{j\bar{x}} \partial_{\bar{x}} + [(\tilde{L}^N)_{\geq 1}\,\tilde{\phi}]\,\partial_{\bar{x}} \\
&= -\sum_{j=1}^{m} \bar{q}_j \bar{r}_j \partial_{\bar{x}} + \sum_{j=1}^{m} \bar{q}_j \partial_{\bar{x}}^{-1} \bar{r}_j \partial_{\bar{x}}^2 + [(\tilde{L}^N)_{\geq 1}\,\tilde{\phi}]\,\partial_{\bar{x}} \\
&= \Big([(\tilde{L}^N)_{\geq 1}\,\tilde{\phi}] + \sum_{j=1}^{m} \tilde{q}_j \widehat{\Omega}(\tilde{r}_j, \tilde{\phi}_x) \Big) \partial_{\bar{x}} + \sum_{j=1}^{m} \bar{q}_j \partial_{\bar{x}}^{-1} \bar{r}_j \partial_{\bar{x}}^2 \,.
\end{aligned}
$$

The dynamics of $\bar{q}_j$, $\tilde{r}_j$ with $j = 1, \ldots, m$ follows from a simple computation. The evolution equation for $\bar{q}_{m+1}$ may then be derived from the Dym dynamics of $\bar{L}$.

□

Proof of theorem 3.1: The result is plausible, when the constraint is regarded as a symmetry reduction $\partial_{t_N} + \partial_{z_2} = 0$, where ∂_{z_2} represents the squared eigenfunction symmetry generated by q_j, r_j. If a suitable z_2-flow of ϕ_i, ψ_i is assumed, then, according to theorem 4.4 in Part I, $z = z_1$, the flow ∂_z commutes with $\partial_{t_N} + \partial_{z_2}$, whence the constraint is invariant under ∂_z. Indeed, an explicit computation yields:

$$
\begin{aligned}
&\Big((L^N)_{<k} - \textstyle\sum_j q_j \partial_x^{-1} r_j \partial_x^k \Big)_z \\
&= \textstyle\sum_i \phi_i \partial_x^{-1} \Big([\partial_x^{-k} (L^N)^*_{\geq k} \partial_x^k \psi_i] - (-1)^k \sum_j \Omega(\psi_i^{(k)}, q_j) r_j \Big)\, \partial_x^k \\
&\quad - \textstyle\sum_i \Big([(L^N)_{\geq k} \phi_i] - \sum_j q_j \widehat{\Omega}(r_j, \phi_i^{(k)}) \Big)\, \partial_x^{-1} \psi_i \partial_x^k \;=\; 0\,.
\end{aligned}
$$

$\square$

Proof of lemma 3.2: For $R = \mathrm{res}(\partial_x^{-1}\psi^{(k)} L^N \phi\, \partial_x^{-1})$ one finds

$$
\begin{aligned}
R_x &\overset{(A.1)}{=} \mathrm{res}(\psi^{(k)} L^N \phi\, \partial_x^{-1} - \partial_x^{-1}\psi^{(k)} L^N \phi) \\
&\overset{(A.3)}{=} \mathrm{res}(\psi^{(k)} L^N \phi\, \partial_x^{-1} - \phi (L^N)^* \psi^{(k)}\, \partial_x^{-1}) \\
&\overset{(A.2)}{=} \psi^{(k)} [L^N \phi] - [(L^N)^* \psi^{(k)}] \phi = (\lambda - \mu)\, \psi^{(k)} \phi\,.
\end{aligned}
$$

Further, with $M = (L^n)_{\geq k}$:

$$
\begin{aligned}
R_{t_n} &= \mathrm{res}(\partial_x^{-1}(-[M^* \psi^{(k)}] L^N \phi + \psi^{(k)} [M, L^N] \phi + \psi^{(k)} L^N [M\phi]) \partial_x^{-1}) \\
&\overset{(A.3)}{=} -\mathrm{res}(\partial_x^{-1} \phi (L^N)^* (M^* \psi^{(k)} - [M^* \psi^{(k)}]) \partial_x^{-1}) \\
&\qquad\qquad\qquad - \mathrm{res}(\partial_x^{-1} \psi^{(k)} L^N (M\phi - [M\phi]) \partial_x^{-1}) \\
&= -\mathrm{res}(\partial_x^{-1} \phi (L^N)^* (M^* \psi^{(k)})_{\geq 1} \partial_x^{-1}) - \mathrm{res}(\partial_x^{-1} \psi^{(k)} L^N (M\phi)_{\geq 1} \partial_x^{-1}) \\
&\overset{(A.4)}{=} -\mathrm{res}(\partial_x^{-1} \phi (L^N)^* (M^* \psi^{(k)}\, \partial_x^{-1})_{\geq 0}) - \mathrm{res}(\partial_x^{-1} \psi^{(k)} L^N (M\phi\, \partial_x^{-1})_{\geq 0}) \\
&\overset{(A.3,A.5)}{=} \mathrm{res}((\partial_x^{-1} \psi^{(k)} M)_{\geq 0} L^N \phi\, \partial_x^{-1}) + \mathrm{res}((\partial_x^{-1} \phi M^*)_{\geq 0} (L^N)^* \psi^{(k)}\, \partial_x^{-1}) \\
&\overset{(A.2)}{=} ((\partial_x^{-1} \psi^{(k)} M)_{\geq 0} L^N \phi)_0 + ((\partial_x^{-1} \phi M^*)_{\geq 0} (L^N)^* \psi^{(k)})_0 \\
&\overset{(A.6)}{=} \lambda\, [(\partial_x^{-1} \psi^{(k)} M)_{\geq 0} \phi] + \mu\, [(\partial_x^{-1} \phi M^*)_{\geq 0} \psi^{(k)}] \\
&\overset{(A.7)}{=} \lambda\, (\partial_x^{-1} \psi^{(k)} M \phi)_0 + \mu\, (\partial_x^{-1} \phi M^* \psi^{(k)})_0 \\
&\overset{(A.2)}{=} \lambda\, \mathrm{res}(\partial_x^{-1} \psi^{(k)} M \phi\, \partial_x^{-1}) + \mu\, \mathrm{res}(\partial_x^{-1} \phi M^* \psi^{(k)}\, \partial_x^{-1}) \\
&\overset{(A.3)}{=} (\lambda - \mu)\, \mathrm{res}(\partial_x^{-1} \psi^{(k)} M \phi\, \partial_x^{-1})\,.
\end{aligned}
$$

Consequently, one may choose the squared eigenfunction potential $\Omega(\psi^{(k)}, \phi) = R/(\lambda - \mu)$. The results for $\widehat{\Omega}(\psi, \phi^{(k)})$ are verified by an analogous computation.

$\square$

Proof of theorem 3.3: Noting $[L^N\phi] = (L^N\phi)_0$ we consider the spectral equation for ϕ in the form $\Delta := \mathrm{res}(\,(L^N\phi - \lambda\phi)\,\partial_x^{-1}) = 0$. One computes

$$\begin{aligned}\Delta_z \;=\; \textstyle\sum_i \mathrm{res}\Big(&\phi_i\partial_x^{-1}\psi_i\partial_x^k L^N\phi\partial_x^{-1} - L^N\phi_i\partial_x^{-1}\psi_i\partial_x^k\phi\partial_x^{-1} \\ &+ (L^N - \lambda)\phi_i\widehat{\Omega}(\psi_i, \phi^{(k)}\partial_x^{-1})\Big)\,,\end{aligned}$$

which, using equation (4.9) in Part I, yields

$$\Delta_z = \textstyle\sum_i \phi_i\left\{\mathrm{res}(\partial_x^{-1}\psi_i\partial_x^k L\phi\partial_x^{-1}) + (-1)^k\lambda_i\Omega(\psi_i^{(k)}, \phi) - \lambda\widehat{\Omega}(\psi_i, \phi^{(k)})\right\}.$$

It can be checked that the spectral equations for ψ_i and ϕ lead to

$$\begin{aligned}&\mathrm{res}(\partial_x^{-1}\psi_i\partial_x^k L\phi\partial_x^{-1}) - \mathrm{res}(\partial_x^{-1}\psi_i\partial_x^k L\partial_x^{-k}\phi^{(k)}\partial_x^{-1}) \\ &= \left\{\begin{array}{cc} 0 & (k=0) \\ \lambda_i\psi_i\phi & (k=1) \\ \lambda_i(\psi_i\phi_x - \psi_{ix}\phi) & (k=2)\end{array}\right\} = \lambda_i\Big(\widehat{\Omega}(\psi_i, \phi^{(k)}) - (-1)^k\Omega(\psi_i^{(k)}, \phi)\Big)\,,\end{aligned}$$

and

$$\begin{aligned}&\mathrm{res}(\partial_x^{-1}\psi_i\partial_x^k L\phi\partial_x^{-1}) - (-1)^k\mathrm{res}(\partial_x^{-1}\psi_i^{(k)} L\phi\partial_x^{-1}) \\ &= \left\{\begin{array}{cc} 0 & (k=0) \\ \lambda\psi_i\phi & (k=1) \\ \lambda(\psi_i\phi_x - \psi_{ix}\phi) & (k=2)\end{array}\right\} = \lambda\Big(\widehat{\Omega}(\psi_i, \phi^{(k)}) - (-1)^k\Omega(\psi_i^{(k)}, \phi)\Big)\,,\end{aligned}$$

whence

$$\begin{aligned}\Delta_z \;&=\; \textstyle\sum_i \phi_i\left\{\mathrm{res}(\partial_x^{-1}\psi_i\partial_x^k L^N\partial_x^{-k}\phi^{(k)}\partial_x^{-1}) - (\lambda - \lambda_i)\,\widehat{\Omega}(\psi_i, \phi^{(k)})\right\} \\ &=\; (-1)^k\,\textstyle\sum_i \phi_i\left\{\,\mathrm{res}(\partial_x^{-1}\psi_i^{(k)} L^N\phi\partial_x^{-1}) - (\lambda - \lambda_i)\,\Omega(\psi_i^{(k)}, \phi)\,\right\} = 0\,.\end{aligned}$$

□