Squared Eigenfunction Symmetries for Soliton Equations: Part I.

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Abstract

Three classes of soliton systems associated with scalar Lax operators are considered. They represent, in turn, the $2+1$ dimensional hierarchies of the KP equation, the modified KP equation and the Dym equation. They are related via gauge transformations and reciprocal links. For each class an algebraic construction of the symmetry transformation generated by products of (adjoint) eigenfunctions is given. The links between the soliton hierarchies are extended to these symmetries.

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1 Introduction

Bäcklund and Reciprocal Transformations represent powerful tools for investigating nonlinear evolution equations. They play a key role in establishing structural properties such as Hamiltonian and/or bi-Hamiltonian structure, symmetry properties, etc. Also, they are a means of finding solutions of nonlinear equations. An extensive bibliography concerning applications of Bäcklund transformations in solving boundary value problems, as well as initial value problems, arising in connection with models in Gas Dynamics, Fluid Mechanics and other areas, is given in [23], and, subsequently, in [22].

For integrable nonlinear equations, arising as the compatibility of two linear equations, Bäcklund and Reciprocal Transformations originate from transformations of the Lax pair. Certain Lax equations are associated with a Lie-algebraic decomposition of the space of pseudo-differential symbols [1, 9, 20]. This framework provides a convenient terminology to discuss the corresponding nonlinear equations and their hierarchies. On the basis of the results established in [12, 18], the links among hierarchies of nonlinear evolution equations in $2+1$ are reconsidered. Section 2 is devoted to notation and a short overview on background facts. In section 3 links among different hierarchies of $2+1$ dimensional equations are considered. Section 4 comprises the introduction of a convenient terminology for squared eigenfunction symmetries associated with these hierarchies. The links are extended to the squared eigenfunction symmetries.

As pointed out in the concluding Section 5, corresponding results exist for the reductions of the 2+1 dimensional hierarchies to soliton equations in 1+1 dimensions. The 1+1 dimensional case is studied in detail in [17].

Proofs of all lemmata and theorems are given in the appendix.

2 Notation and background

Consider the algebra g of pseudo-differential symbols and its decompositions

$$g = \Big\{ \sum_{-\infty < i \ll \infty} u_i \, \partial_x^i \Big\} = \Big\{ \sum_{i \geq k} u_i \, \partial_x^i \Big\} \oplus \Big\{ \sum_{i < k} u_i \, \partial_x^i \Big\} =: g_{\geq k} \oplus g_{<k} \qquad (2.1)$$

into differential orders $\geq k$ and $< k$, where

$$\partial_x^i u = u \, \partial_x^i \; + \sum_{j=1}^{\infty} \frac{i(i-1)\cdots(i-j+1)}{j!} \; u^{(j)} \, \partial_x^{i-j} \; , \quad u^{(j)} \equiv u_{\underbrace{x \ldots x}_{j}} \equiv \frac{\partial^j u}{\partial x^j} \, .$$

In particular, $\partial_x^{-1} u = u \, \partial_x^{-1} - u_x \, \partial_x^{-2} + u_{xx} \, \partial_x^{-3} - \cdots$. The coefficients u_i are scalar functions of $x \equiv t_1$ and some further "time" parameters $t_2, t_3, \ldots$. For $L = \sum_i u_i \, \partial_x^i \in g$ the projections to $g_{\geq k}$ and $g_{<k}$ are denoted by

$$L_{\geq k} = \sum_{i \geq k} u_i \partial_x^i \; , \quad L_{<k} = \sum_{i<k} u_i \partial_x^i \, ,$$

further we introduce the zero-order term and the residue

$$L_0 = u_0 \; , \quad \mathrm{res}(L) = u_{-1} \; .$$

In (2.1) the cases $k = 0, 1, 2$ are distinguished by the fact that $g_{\geq k}$ and $g_{<k}$ define sub-Lie algebras of g: $[g_{\geq k}, g_{\geq k}] \subset g_{\geq k}$ and $[g_{<k}, g_{<k}] \subset g_{<k}$. Hence, in

these cases, the decompositions (2.1) define classical r-matrices [25]

$$r_k : \ L \ \rightarrow \ L_{\geq k} - L_{<k}$$

on g and give rise to three hierarchies of integrable Lax equations [2, 10, 18] which will be reviewed in section 3.

Remark 2.1: *For a differential operator $M \in g_{\geq 0}$ the symbol $[M\phi]$ denotes the action of M on the function ϕ. This is not to be confused with $M\phi$, denoting the composition of M with the multiplication operator given by ϕ. We note $[M\phi] = (M\phi)_0$.*

The variational adjoint of an operator in g is given by

$$(\textstyle\sum_i u_i \, \partial_x^i)^* \ = \ \textstyle\sum_i (-1)^i \, \partial_x^i u_i \ ,$$

whence for any differential operator $M \in g_{\geq 0}$ and all functions ϕ, ψ the expression $\psi[M\phi] - [M^*\psi]\phi = \omega_x$ is a perfect x-derivative of some field ω. In the context of pseudo-differential symbols ω is conveniently expressed by

$$\omega = \mathrm{res}(\partial_x^{-1} \psi M \phi \, \partial_x^{-1}) \ .$$

One has the following lemma, which defines the "squared eigenfunction potential":

Lemma 2.2: *For any differential operator $M \in g_{\geq 0}$ and any pair of func-*

tions ϕ, ψ solving the two adjoint equations

$$\phi_t = [M\phi] \ , \quad \psi_t = -[M^*\psi] \tag{2.2}$$

there exists a potential $\Omega = \Omega(\psi, \phi)$ defined by the compatible equations

$$\Omega_x = \psi\phi \ , \quad \Omega_t = \mathrm{res}(\partial_x^{-1}\psi M\phi\,\partial_x^{-1}) \,.$$

This potential, to be regarded as a bilinear functional of ϕ and ψ, is well-defined up to an additive constant. We remark that the evolution parameter t may actually represent a collection of "times" t_n introduced by $\phi_{t_n} = [M_n\phi]$ and $\psi_{t_n} = -[M_n^*\psi]$, where $M_n \in g_{\geq 0}$ is a family of differential operators satisfying the "zero curvature" conditions $M_{n_1 t_{n_2}} - M_{n_2 t_{n_1}} = [M_{n_2}, M_{n_1}]$ granting the compatibility of the linear flows. Then all equations

$$\Omega_x = \psi\phi \ , \quad \Omega_{t_n} = \mathrm{res}(\partial_x^{-1}\psi M_n\phi\,\partial_x^{-1})$$

are compatible.

3 Soliton hierarchies and their links

In this section we review established results on the algebraic construction of the 2+1 dimensional hierarchies of the KP, the mKP and the Dym equations as well as the links which relate them.

Application of Bäcklund and reciprocal transformations to interconnect nonlinear evolution equations has its origin in the work by Rogers and Wong

[24]. Subsequently, links among nonlinear evolution equations have been shown to represent a powerful tool to reveal interesting properties of whole hierarchies of nonlinear evolution equations [4, 8]. Specifically, equations connected to each other via Bäcklund Charts admit symmetries, bi-Hamiltonian structures, recursion operators etc., which are related [7]. In particular, the transformations relating the Dym equation and the KdV equation [24] allow to recognize the interesting symmetry structures of the Dym equation [8]. The corresponding hierarchy [3] has been proved to exhibit an invariance under a reciprocal transformation, which induces a generic auto-Bäcklund transformation for the KdV hierarchy. A reciprocal connection between the KP equation and the 2+1 dimensional Dym equation, as introduced by Konopelchenko and Dubrovsky [11] was noted in [21]. A review on recent results in 2+1 dimensions is given by Konopelchenko and Rogers [13].

Using the pseudo-differential approach, gauge and reciprocal links among hierarchies of nonlinear evolution equations in $2+1$ dimensions have been established in [18]. In particular, they provide novel auto-Bäcklund transformations for the Dym hierarchy.

The algebraic construction of the KP hierarchy and its relatives is based on the following simple observation [2, 10, 18]:

Theorem 3.1: *Let $k = 0, 1$ or 2. The equations of the Lax hierarchy*

$$L_{t_n} = [(L^n)_{\geq k}, L]$$

with

$$\begin{aligned} k = 0: \quad L_{\mathrm{KP}} &= \partial_x + u\,\partial_x^{-1} + u_2\,\partial_x^{-2} + u_3\,\partial_x^{-3} + \cdots \\ k = 1: \quad L_{\mathrm{mKP}} &= \partial_x + v + v_1\,\partial_x^{-1} + v_2\,\partial_x^{-2} + \cdots \\ k = 2: \quad L_{\mathrm{Dym}} &= w\,\partial_x + w_0 + w_1\,\partial_x^{-1} + w_2\,\partial_x^{-2} + \cdots \end{aligned} \tag{3.1}$$

commute.

Following the r-matrix scheme [25] this result is a simple consequence of the Lie algebra decomposition (2.1) with $k = 0, 1$ or 2.

Each Lax equation represents an evolution equation for the infinite vector $(u, u_2, \ldots)$, $(v, v_1, \ldots)$ and $(w, w_0, \ldots)$, respectively. Equations in closed form can be extracted for the "prime" fields u, v, w. The equations resulting from $L_y = [(L^2)_{\geq k}, L]$ (with $y = t_2$) are used to express all fields in terms of u, v or w, respectively, and their y-derivatives. In particular, the equations defining the "auxiliary" fields u_2, v_1 and w_0, respectively, are computed as

$$\begin{aligned} k = 0: \quad u_y &= u_{xx} + 2u_{2x} \\ k = 1: \quad v_y &= v_{xx} + 2vv_x + 2v_{1x} \\ k = 2: \quad w_y &= w^2 w_{xx} + 2w^2 w_{0x}. \end{aligned} \tag{3.2}$$

The Lax equations for $n \geq 3$ represent equations in t_n, x and y for the fields u, v and w. In particular, the equations associated with $n = 3$ are

$$\begin{aligned} k = 0: \quad 4u_{t_3 x} &= (u_{xxx} + 12uu_x)_x + 3u_{yy} && \text{(KP)} \\ k = 1: \quad 4v_{t_3 x} &= (v_{xxx} - 6v^2 v_x)_x + 3v_{yy} + 6v_x v_y + 6v_{xx} D^{-1}(v_y) && \text{(mKP)} \\ k = 2: \quad 4w_{t_3} &= w^3 w_{xxx} + \frac{1}{2}\left(w^2 D^{-1}\left(\frac{w_y}{w^2}\right)\right)_y, && \text{(Dym)} \end{aligned}$$

i.e., the KP equation, the modified KP equation and the 2+1 dimensional Dym equation. The integration $D^{-1} = \int^x$ arises from the elimination of the auxiliary fields via (3.2). For the derivation of these equations one computes

$$
\begin{array}{llll}
k=0: & (L^2_{\mathrm{KP}})_{\geq 0} = \partial_x^2 + 2u, & (L^3_{\mathrm{KP}})_{\geq 0} = \partial_x^3 + 3u\,\partial_x + 3(u_x + u_2) \\
k=1: & (L^2_{\mathrm{mKP}})_{\geq 1} = \partial_x^2 + 2v\,\partial_x, & (L^3_{\mathrm{mKP}})_{\geq 1} = \partial_x^3 + 3v\,\partial_x^2 + 3(v_x + v^2 + v_1) \\
k=2: & (L^2_{\mathrm{Dym}})_{\geq 2} = w^2\,\partial_x^2, & (L^3_{\mathrm{Dym}})_{\geq 2} = w^3\,\partial_x^3 + 3w^2(w_x + w_0)\,\partial_x^2.
\end{array}
$$

Eigenfunctions are defined as solutions of the evolution equations $\phi_{t_n} = [(L^n)_{\geq k}\phi]$, which for $n = 2, 3$ yield

$$
\begin{array}{lll}
k=0: & \phi_y = \phi_{xx} + 2u\phi, & \phi_{t_3} = \phi_{xxx} + 3u\phi_x + 3(u_x + u_2)\phi \\
k=1: & \phi_y = \phi_{xx} + 2v\phi_x, & \phi_{t_3} = \phi_{xxx} + 3v\phi_{xx} + 3(v_x + v^2 + v_1)\phi_x \\
k=2: & \phi_y = w^2\phi_{xx}, & \phi_{t_3} = w^3\phi_{xxx} + 3w^2(w_x + w_0)\phi_{xx}.
\end{array}
\tag{3.3}
$$

The three hierarchies are linked through Darboux type transformations generated by functions satisfying the eigenfunction dynamics:

Theorem 3.2: [12, 18]

a) (KP $\rightarrow$ mKP): If $L_{t_n} = [(L^n)_{\geq 0}, L]$ and $\phi_{t_n} = [(L^n)_{\geq 0}\phi]$, then $\tilde{L} = \phi^{-1}L\phi$ satisfies $\tilde{L}_{t_n} = [(\tilde{L}^n)_{\geq 1}, \tilde{L}]$.

b) (mKP $\rightarrow$ Dym): If $\tilde{L}_{t_n} = [(\tilde{L}^n)_{\geq 1}, \tilde{L}]$ and $\tilde{\phi}_{t_n} = [(\tilde{L}^n)_{\geq 1}\tilde{\phi}]$, then, after the transformation $\bar{x} = \tilde{\phi}(x, t_n)$ and $\bar{t}_n = t_n$ of the independent variables, the operator $\bar{L} = \tilde{L}$ satisfies $\bar{L}_{\bar{t}_n} = [(\bar{L}^n)_{\overline{\geq 2}}, \bar{L}]$, where the projection $(.)_{\overline{\geq 2}}$ refers to powers of $\partial_{\bar{x}} = \partial/\partial\bar{x}$.

In $b)$ the change of variable $x \rightarrow \bar{x} = \tilde{\phi}(x, t_n)$ induces a transformation $\partial_x \rightarrow \partial_{\bar{x}} = \tilde{\phi}_x^{-1}\partial_x$, $\partial_x^{-1} \rightarrow {\partial_{\bar{x}}}^{-1} = \partial_x^{-1}\tilde{\phi}_x$ of the pseudo-differential symbols as well as a transformation $\tilde{w}(x,t_n) \to \bar{w}(\bar{x},\bar{t}_n) = \tilde{w}(x,t_n)$ of functions. In particular, this link maps $\tilde{L} = \sum_i v_i \partial_x{}^i$ to $\bar{L} = \sum_i w_i \partial_{\bar{x}}{}^i = \sum_i v_i (\tilde{\phi}_x \partial_{\bar{x}})^i$. In short, the transformation b) is a pure change of the independent variables: the new Lax operator $\bar{L}$ coincides with the the original $\tilde{L}$, which has to be rewritten in powers of the new symbol $\partial_{\bar{x}} = \partial/\partial\bar{x}$ associated with the new "space" variable $\bar{x}$.

The transformation a) of this theorem applied to the Lax operator of the KP hierarchy leads to

$$L_{\mathrm{mKP}} = \phi^{-1} L_{\mathrm{KP}} \phi = \phi^{-1}(\partial_x + u\,\partial_x^{-1} + \cdots)\phi = \partial_x + \phi^{-1}\phi_x + u\,\partial_x^{-1} + \cdots ,$$

whence comparison with (3.1,$k=1$) leads to the transformation $u \to v = \phi^{-1}\phi_x$, where ϕ is an eigenfunction of the KP hierarchy satisfying (3.3,$k=0$). The inverse map $v \to u = v_1$ with v_1 defined by (3.2,$k=1$) is the standard Miura transformation

$$(2u + v_x + v^2)_x = v_y$$

between the KP hierarchy and the modified KP hierarchy.

Similarly, the transformation b) between the mKP and the Dym hierarchies is obtained by

$$L_{\mathrm{Dym}} = L_{\mathrm{mKP}} = \partial_x + v + \cdots = \tilde{\phi}_x \partial_{\bar{x}} + v + \cdots ,$$

whence comparison with (3.1,$k=2$) leads to the transformation $v \to w = \tilde{\phi}_x$, $x \to \bar{x} = \tilde{\phi}$, $y \to \bar{y} = y$, where $\tilde{\phi}$ is any eigenfunction of the modified KP hierarchy satisfying (3.3,$k=1$). The inverse map is given by $w \to v = w_0$ with w_0 defined by (3.2,$k=2$) in the variables $\bar{x}$, $\bar{y}$, i.e.,

$$2v_x = \frac{w_{\bar{y}}}{w} - w w_{\bar{x}\bar{x}}\,, \quad \frac{\partial \bar{x}}{\partial x} = w\,, \quad \bar{y} = y\ .$$

A 1+1-dimensional reciprocal link of such kind was first derived in the context of Painlevé expansions [26, 27].

4 Squared eigenfunction symmetries and their links

In this section we formulate the squared eigenfunction symmetries associated with the 2+1 dimensional hierarchies considered in the previous section. The links of section 3 are extended to these symmetries.

In $2+1$ dimensions eigenfunctions are defined as solutions of the linear systems $\phi_{t_n} = [(L^n)_{\geq k}\phi]$, where the equation associated with $t_2 = y$ is regarded as the scattering problem used to solve the corresponding nonlinear equations via Inverse Scattering techniques [6]. The fact that products of eigenfunctions and adjoint eigenfunctions can be regarded as infinitesimal generators of symmetry transformations is crucial for such considerations. In this section we formulate these symmetries algebraically for the three hierarchies $k = 0, 1, 2$ considered in the previous section.

For the cases $k = 1$ and 2 it turns out that integrated versions of the usual adjoint linear problems are more appropriate. One observes that the operators $\partial_x^{-k}(L^n)^*_{\geq k}\partial_x^k = (\partial_x^k(L^n)_{\geq k}\partial_x^{-k})^*$ are again *differential* operators. The "adjoint" linear evolution equations to be considered are, in fact, associated with these operators rather than with the usual adjoints $(L^n)^*_{\geq k}$.

With $L = L_{\rm KP}, L = L_{\rm mKP}$ or $L = L_{\rm Dym}$, respectively, we shall define eigenfunctions and "adjoint eigenfunctions" as solutions of the linear problems

$$\phi_{t_n} = [(L^n)_{\geq k}\phi] \ , \quad \psi_{t_n} = -[\partial_x^{-k}(L^n)^*_{\geq k}\partial_x^k\psi] \, . \tag{4.1}$$

We remark that for $k = 1$ and 2, respectively, the fields ψ_x and ψ_{xx}, respectively, satisfy the usual adjoint problems given by $(L^n)^*_{\geq k}$, whence ψ defined by (4.1) should be regarded as integrated version of a proper adjoint eigenfunction. However, for simplicity, we will refer to solutions of (4.1) as (adjoint) eigenfunctions in the following.

Squared eigenfunctions, i.e., products of eigenfunctions and their adjoint generate a one parameter symmetry group of the Lax equations. This fact is of fundamental importance in various aspects of KP theory. We will denote the corresponding group parameter by z and characterize this symmetry via its infinitesimal generator, which is given as an evolution equation in Lax form [19]:

Theorem 4.1: *Let $k = 0, 1$ or 2. Let L satisfy the hierarchy of Lax equations $L_{t_n} = [(L^n)_{\geq k}, L]$, let $\phi_1, \ldots, \phi_m$ and $\psi_1, \ldots, \psi_m$ be (adjoint) eigenfunctions* *satisfying (4.1). Then the flow generated by*

$$L_z = \left[\sum_{i=1}^{m} \phi_i \partial_x^{-1} \psi_i \, \partial_x^k \, , \; L \right] \tag{4.2}$$

commutes with the Lax hierarchy.

For the three soliton hierarchies of section 3 these symmetries imply the following z-evolutions of the highest non-trivial coefficients in the Lax operators (3.1). From the highest orders of (4.2) with $L = L_{\rm KP}, L_{\rm mKP}, L_{\rm Dym}$, respectively, one obtains

$$\begin{aligned}
k = 0: \quad & u_z = -\textstyle\sum_i (\phi_i \psi_i)_x \, , \; \phi_{iy} = \phi_{ixx} + 2u\phi_i \, , \; \psi_{iy} = -\psi_{ixx} - 2u\psi_i \\
k = 1: \quad & v_z = -\textstyle\sum_i (\phi_i \psi_i)_x \, , \; \phi_{iy} = \phi_{ixx} + 2v\phi_{ix} \, , \; \psi_{iy} = -\psi_{ixx} + 2v\psi_{ix} \\
k = 2: \quad & w_z = \textstyle\sum_i (\phi_i \psi_i w_x - (\phi_i \psi_i)_x w) \, , \; \phi_{iy} = w^2 \phi_{ixx} \, , \; \psi_{iy} = -w^2 \psi_{ixx} \, ,
\end{aligned}$$

where the $y = t_2$ evolutions of the (adjoint) eigenfunctions are generated by $(L^2)_{\geq k}$. For the multicomponent version of the (modified) KP hierarchy these are interesting 2+1 dimensional integrable systems of their own [19], they include the Darboux system and a sine-Gordon equation with the two space variables occurring on the same footing [14].

Remark 4.2: *The existence of the squared eigenfunction symmetry does not depend on the fact that the nonlinear soliton equations are constructed from Lax equations* $L_{t_n} = [M_n, L]$ *with* $M_n = (L^n)_{\geq k}$. *Generally, one may consider soliton equations in* $2 + 1$ *dimensions given by a zero curvature* *representation*

$$\Delta := M_{n_2 t_{n_1}} - M_{n_1 t_{n_2}} + [M_{n_2}, M_{n_1}] = 0 \tag{4.3}$$

with differential operators M_n*, representing the compatibility condition of the linear problems* $\phi_{t_n} = [M_n \phi]$ *and* $\psi_{t_n} = -[M_n^* \psi]$*, respectively, for different choices of* n*. With (adjoint) eigenfunctions* ϕ_i*,* ψ_i *the corresponding squared eigenfunction symmetry is characterized by (see* (A.*10,* $k=0$*))*

$$M_{nz} = [M', M_n]_{\geq 0}\,, \quad M' = \sum_i \phi_i \partial_x^{-1} \psi_i\,, \tag{4.4}$$

which is compatible with the zero curvature representation (4.3). This follows from the pseudo-differential zero curvature equation (see (A.*12))*

$$M_{nz} - M'_{t_n} = [M', M_n]$$

leading to $\Delta_z = [\, M'\,, \Delta\,] = 0$*, whence (4.3) is invariant under the* z*-flow.*

If the operators in (4.3) do not have terms of order 0 *(or* 0 *and* 1*, respectively), then (4.4) cannot be imposed because the right hand side includes these orders. The squared eigenfunction symmetry then has to be modified as indicated by the cases* $k = 1, 2$ *of the previous considerations.*

The Lax equations and their adjoints

$$L_{t_n} = [(L^n)_{\geq k}, L]\ , \quad \partial_x^{-k} L^*_{t_n} \partial_x^k = -[\partial_x^{-k} (L^n)^*_{\geq k} \partial_x^k, \partial_x^{-k} L^* \partial_x^k]$$

are the compatibility conditions of the linear equations $\phi_{t_n} = [(L^n)_{\geq k} \phi]$ and their adjoints $\psi_{t_n} = -[\partial_x^{-k} (L^n)^*_{\geq k} \partial_x^k \psi]$ for different choices of n. Also the squared eigenfunction symmetry may be regarded as the compatibility condition of linear equations, where, however, the z-evolution is not a differential equation, but involves the squared eigenfunction potential of lemma 2.2. Formally, these linear equations are $\phi_z = \sum_i \phi_i [\![\partial_x^{-1} \psi_i \partial_x^k \phi]\!]$ and

$$\psi_z = -[\![\partial_x^{-k} (\sum_i \phi_i \partial_x^{-1} \psi_i \partial_x^k)^* \partial_x^k \psi]\!] = (-1)^k \sum_i \psi_i [\![\partial_x^{-1} \phi_i \partial_x^k \psi]\!] \, ,$$

where $[\![\, . \,]\!]$ means "application of an operator to a function". However, the pseudo-differential symbol ∂_x^{-1} is not supposed to act on ordinary functions. Heuristically, $[\![\partial_x^{-1} \, . \,]\!]$ has to be replaced by a squared eigenfunction potential, wich gives ∂_x^{-1} the effect of an ordinary integration and, at the same time, fixes the time dependence of the integration constant:

Lemma 4.3: *The squared eigenfunction symmetry of theorem 4.1 is the compatibility condition of the linear problems*

$$\phi_{t_n} = [(L^n)_{\geq k} \phi] \, , \quad \phi_z = \sum_i \phi_i \widehat{\Omega}(\psi_i, \phi^{(k)})$$

and

$$\psi_{t_n} = -[\partial_x^{-k} (L^n)^*_{\geq k} \partial_x^k \psi] \, , \quad \psi_z = (-1)^k \sum_i \Omega(\psi^{(k)}, \phi_i) \psi_i \, , \tag{4.5}$$

respectively. Here $\Omega(\, . \, , . \,)$ is the potential of lemma 2.2, i.e.,

$$\Omega(\psi^{(k)}, \phi)_x = \psi^{(k)} \phi \, , \quad \Omega(\psi^{(k)}, \phi)_{t_n} = \mathrm{res}(\partial_x^{-1} \psi^{(k)} (L^n)_{\geq k} \phi \, \partial_x^{-1}) \, , \tag{4.6}$$

where, in turn, $\psi^{(k)}$ denotes ψ, ψ_x or ψ_{xx} for $k = 0, 1$ or 2. The potential

$\widehat{\Omega}(\,.\,,.)$ is defined by

$$\widehat{\Omega}(\psi,\phi^{(k)}) = \begin{cases} \Omega(\psi,\phi) & (k=0) \\ -\Omega(\psi_x,\phi)+\psi\phi & (k=1) \\ \Omega(\psi_{xx},\phi)-\psi_x\phi+\psi\phi_x & (k=2). \end{cases} \tag{4.7}$$

It satisfies the compatible equations

$$\widehat{\Omega}(\psi,\phi^{(k)})_x = \psi\phi^{(k)}\ , \quad \widehat{\Omega}(\psi,\phi^{(k)})_{t_n} = \mathrm{res}(\partial_x^{-1}\psi\partial_x^k(L^n)_{\geq k}\partial_x^{-k}\phi^{(k)}\,\partial_x^{-1})\,, \tag{4.8}$$

where, in turn, $\phi^{(k)}$ denotes ϕ,ϕ_x or ϕ_{xx} for $k=0,1$ or 2.

We note the operator identity

$$\partial_x^{-1}\psi\partial_x^k\phi\,\partial_x^{-1} = \widehat{\Omega}(\psi,\phi^{(k)})\,\partial_x^{-1} - (-1)^k\partial_x^{-1}\Omega(\psi^{(k)},\phi) + \frac{k(k-1)}{2}\,\psi\phi\,. \tag{4.9}$$

We now consider the commutativity of two squared eigenfunction symmetries generated by different pairs of (adjoint) eigenfunctions ψ_i,ϕ_i ($i=1,\ldots,m_1$) and r_j,q_j ($j=1,\ldots,m_2$), respectively:

Theorem 4.4: *Let ϕ_i,ψ_i ($i=1,\ldots,m_1$) and q_j,r_j ($j=1,\ldots,m_2$) satisfy*

$$\begin{aligned} \phi_{it_n} &= [(L^n)_{\geq k}\phi_i]\,, & \psi_{it_n} &= -[\partial_x^{-k}(L^n)^*_{\geq k}\partial_x^k\psi_i] \\ q_{jt_n} &= [(L^n)_{\geq k}q_j]\,, & r_{jt_n} &= -[\partial_x^{-k}(L^n)^*_{\geq k}\partial_x^k r_j] \end{aligned}$$

and

$$\begin{aligned} \phi_{iz_2} &= \textstyle\sum_j q_j\widehat{\Omega}(r_j,\phi_i)\,, & \psi_{iz_2} &= (-1)^k\textstyle\sum_j\Omega(\psi_i,q_j)r_j \\ q_{jz_1} &= \textstyle\sum_i \phi_i\widehat{\Omega}(\psi_i,q_j)\,, & r_{jz_1} &= (-1)^k\textstyle\sum_i\Omega(r_j,\phi_i)\psi_i \end{aligned}$$

with $\Omega(\,.\,,.)$ and $\widehat{\Omega}(\,.\,,.)$ as in lemma 4.3. Then

$$M' = \textstyle\sum_i \phi_i\partial_x^{-1}\psi_i\partial_x^k \quad \text{and} \quad M'' = \textstyle\sum_j q_j\partial_x^{-1}r_j\partial_x^k$$

satisfy the zero curvature equation $M'_{z_2} - M''_{z_1} = [M'', M']$, *whence the Lax equations* $L_{z_1} = [M', L]$ *and* $L_{z_2} = [M'', L]$ *commute.*

Remark 4.5: *The zero curvature equation* $M'_{z_2} - M''_{z_1} = [M'', M']$ *indicates the compatibility of the linear equation*

$$\phi_{z_1} = \sum_i \phi_i \widehat{\Omega}(\psi_i, \phi) \quad \text{and} \quad \phi_{z_2} = \sum_j r_j \widehat{\Omega}(q_j, \phi).$$

Indeed, these equations commute, provided the z*-dependence of the potentials is chosen according to*

$$\widehat{\Omega}(r_j, \phi)_{z_1} = \sum_i \widehat{\Omega}(r_j, \phi_i)\widehat{\Omega}(\psi_i, \phi)\,, \quad \widehat{\Omega}(\psi_i, \phi)_{z_2} = \sum_j \widehat{\Omega}(\psi_i, q_j)\widehat{\Omega}(r_j, \phi)\,,$$

equivalent to

$$\Omega(r_j, \phi)_{z_1} = (-1)^k \sum_i \Omega(r_j, \phi_i)\Omega(\psi_i, \phi)\,,$$
$$\Omega(\psi_i, \phi)_{z_2} = (-1)^k \sum_j \Omega(\psi_i, q_j)\Omega(r_j, \phi)\,.$$

It may be verified that these evolutions are compatible with the defining equations (4.6) and (4.7)/(4.8).

The following theorem extends the links between the three hierarchies as given by theorem 3.2 to the squared eigenfunction symmetries:

Theorem 4.6: *Let* $\Omega(\,.\,,.)$, $\widehat{\Omega}(\,.\,,.)$ *be the potentials of lemma 4.3.*

a) (KP $\rightarrow$ *mKP): Let* L *be a solution of the KP hierarchy* $L_{t_n} = [(L^n)_{\geq 0}, L]$ *with (adjoint) eigenfunctions* $\phi_1, \ldots, \phi_m$, $\psi_1, \ldots, \psi_m$ *satisfying (4.1,* $k = 0$*) and generating the squared eigenfunction symmetry*

$L_z = [\sum_i \phi_i \partial_x^{-1} \psi_i, L]$. *Let* ϕ *be a further eigenfunction satisfying* $\phi_z = \sum_i \phi_i \widehat{\Omega}(\psi_i, \phi)$. *Then, according to theorem 3.2.a),* $\tilde{L} = \phi^{-1} L \phi$ *satisfies the modified KP hierarchy* $\tilde{L}_{t_n} = [(\tilde{L}^n)_{\geq 1}, \tilde{L}]$ *and*

$$\tilde{\phi}_i = \phi^{-1} \phi_i \ , \quad \tilde{\psi}_i = -\widehat{\Omega}(\psi_i, \phi) = -\Omega(\psi_i, \phi)$$

are eigenfunctions of $\tilde{L}$*:*

$$\tilde{\phi}_{it_n} = [(\tilde{L}^n)_{\geq 1} \tilde{\phi}_i] \ , \quad \tilde{\psi}_{it_n} = -[\partial_x^{-1} (\tilde{L}^n)^*_{\geq 1} \partial_x \, \tilde{\psi}_i] \ .$$

Further, $\tilde{L}$ *satisfies* $\tilde{L}_z = [\sum_i \tilde{\phi}_i \partial_x^{-1} \tilde{\psi}_i \partial_x \, , \tilde{L}\,]$.

b) (mKP $\to$ *Dym): Let* $\tilde{L}$ *be a solution of the modified KP hierarchy* $\tilde{L}_{t_n} = [(\tilde{L}^n)_{\geq 1}, \tilde{L}]$ *with (adjoint) eigenfunctions* $\tilde{\phi}_1, \ldots, \tilde{\phi}_m$, $\tilde{\psi}_1, \ldots, \tilde{\psi}_m$ *satisfying (4.1,* $k = 1$, L *replaced by* $\tilde{L}$*) and generating the squared eigenfunction symmetry* $\tilde{L}_z = [\sum_i \tilde{\phi}_i \partial_x^{-1} \tilde{\psi}_i \partial_x \, , \tilde{L}\,]$. *Let* $\tilde{\phi}$ *be a further eigenfunction satisfying* $\tilde{\phi}_z = \sum_i \tilde{\phi}_i \widehat{\Omega}(\tilde{\psi}_i, \tilde{\phi}_x) = \sum_i \tilde{\phi}_i (\tilde{\psi}_i \tilde{\phi} - \Omega(\tilde{\psi}_{ix}, \tilde{\phi}))$. *Then, according to theorem 3.2.b), after the transformation* $\bar{x} = \tilde{\phi}$, $\bar{t}_n = t_n$, $\bar{z} = z$, *the operator* $\bar{L} = L$ *satisfies the Dym hierarchy* $\bar{L}_{\bar{t}_n} = [(\bar{L}^n)_{\overline{\geq 2}} \, , \bar{L}]$ *and*

$$\bar{\phi}_i = \tilde{\phi}_i \ , \quad \bar{\psi}_i = -\widehat{\Omega}(\tilde{\psi}_i, \tilde{\phi}_x) = \Omega(\tilde{\psi}_{ix}, \tilde{\phi}) - \tilde{\psi}_i \tilde{\phi}$$

are eigenfunctions of $\bar{L}$*:*

$$\bar{\phi}_{i\bar{t}_n} = [(\bar{L}^n)_{\overline{\geq 2}} \, \bar{\phi}_i] \ , \quad \bar{\psi}_{i\bar{t}_n} = -[\partial_{\bar{x}}^{-2} (\bar{L}^n)^{\overline{*}}_{\overline{\geq 2}} \, \partial_{\bar{x}}^2 \bar{\psi}_i] \ ,$$

where the transposition $\overline{*}$ *refers to* $(w \partial_{\bar{x}}^i)^{\overline{*}} = (-1)^i \partial_{\bar{x}}^i w$. *Further,* $\bar{L}$ *satisfies* $\bar{L}_{\bar{z}} = [\sum_i \bar{\phi}_i \partial_{\bar{x}}^{-1} \bar{\psi}_i \partial_{\bar{x}}^2, \bar{L}\,]$.

5 Conclusions

Here we have been concentrating our investigations on $2+1$ dimensional soliton systems. On adopting Sato's approach it turns out that Bäcklund as well as reciprocal links can be easily represented for whole hierarchies. Squared eigenfunction symmetries, according to their definition in the previous sections, represent further integrable equations augmenting the hierarchies. They also arise as the compatibility conditions of several linear problems. The links between the soliton hierarchies extend naturally to the corresponding squared eigenfunction symmetries.

Various 1+1 dimensional soliton equations can be obtained from the 2+1 dimensional hierarchies via suitable constraints [5, 15, 16, 28]. The approach proposed here also provides a unifying picture of the Bäcklund and reciprocal links for these reductions. Details of the 1+1 dimensional case are discussed in [17]).

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Appendix

In the computations following rules are used which hold for arbitrary operators $A, B \in g$ and arbitrary functions f:

$$
\begin{aligned}
\mathrm{res}(A)_x &= \mathrm{res}(\partial_x A - A\partial_x) && \text{(A.1)}\\
\mathrm{res}(A\partial_x^{-1}) &= A_0 && \text{(A.2)}\\
\mathrm{res}(A) &= -\mathrm{res}(A^*) && \text{(A.3)}\\
A_{\geq 1} &= (A\partial_x^{-1})_{\geq 0} && \text{(A.4)}\\
(A_{\geq 0})^* &= (A^*)_{\geq 0} && \text{(A.5)}\\
[A_{\geq 0}B_{\geq 0}f] &= [A_{\geq 0}[B_{\geq 0}f]] && \text{(A.6)}\\
[A_{\geq 0}f] &= (Af)_0 && \text{(A.7)}\\
(A_{\geq 0}\partial_x^{-1})_{<0} &= A_0\partial_x^{-1} && \text{(A.8)}\\
(\partial_x^{-1}A_{\geq 0})_{<0} &= \partial_x^{-1}(A^*)_0\,. && \text{(A.9)}
\end{aligned}
$$

They are all checked by simple and straightforward computations using the representation

$$A = \cdots + A_1\partial_x + A_0 + A_{-1}\partial_x^{-1} + \cdots \ .$$

Proof of lemma 2.2: Using the formulae above one finds

$$\mathrm{res}(\partial_x^{-1}\psi M\phi\,\partial_x^{-1})_x \stackrel{(A.1)}{=} \mathrm{res}(\psi M\phi\,\partial_x^{-1}) - \mathrm{res}(\partial_x^{-1}\psi M\phi)$$

$$\stackrel{(A.2,A.3)}{=} \psi[M\phi] - \mathrm{res}(\phi M^*\psi\,\partial_x^{-1}) \stackrel{(A.2)}{=} \psi[M\phi] - [M^*\psi]\phi = (\psi\phi)_t\,,$$

i.e., $\Omega_{tx} = \Omega_{xt}$, whence the two equations defining Ω are compatible.

$\square$

Proof of theorem 4.1: We put $M_n = (L^n)_{\geq k}$ and $M' = \sum_i \phi_i\partial_x^{-1}\psi_i\,\partial_x^k$ for convenience. The Lax equation $L_z = [M', L]$ leads to $L_z^n = [M', L^n]$, whence projection to differential orders $\geq k$ yields $M_{nz} = [M', L^n]_{\geq k}$. Since $(L^n)_{<k}$ does not contribute (note that $M' \in g_{<k}$ and $[g_{<k}, g_{<k}] \subset g_{<k}$ for $k = 0, 1, 2$), we can replace L^n by M_n and obtain

$$M_{nz} = [M', M_n]_{\geq k}\,. \tag{A.10}$$

On the other hand, from

$$\begin{aligned} & \textstyle\sum_i(M_n\phi_i\partial_x^{-1}\psi_i\,\partial_x^k)_{<k} - \sum_i(\phi_i\partial_x^{-1}\psi_i\,\partial_x^k M_n)_{<k} \\ = \quad & \textstyle\sum_j(M_n\phi_i\partial_x^{-1})_{<0}\,\psi_i\,\partial_x^k - \sum_j \phi_i(\partial_x^{-1}\psi_i\,\partial_x^k M_n\partial_x^{-k})_{<0}\,\partial_x^k \\ \stackrel{(A.8,A.9)}{=} \quad & \textstyle\sum_i[M_n\phi_i]\partial_x^{-1}\psi_i\,\partial_x^k - \sum_i \phi_i\partial_x^{-1}[\partial_x^{-k}M_n^*\partial_x^k\psi_i]\,\partial_x^k \end{aligned}$$

one concludes

$$[M_n, M']_{<k} = M'_{t_n} \,. \tag{A.11}$$

This leads to the pseudo-differential "zero curvature" equation

$$M_{nz} - M'_{t_n} = [M', M_n] \,, \tag{A.12}$$

where (A.10) and (A.11), in turn, represent the differential orders $\geq k$ and $<$ k of (A.12). This establishes the commutativity of the squared eigenfunction flow and the Lax hierarchy:

$$L_{zt_n} - L_{t_n z} = [M', L]_{t_n} - [M_n, L]_z = [\, M'_{t_n} - M_{nz} - [M_n, M'] \,,\, L\,] = 0 \,.$$

$\square$

Proof of lemma 4.3: The compatibility of (4.6) was verified in lemma 2.2. A straightforward computation[1] shows that(4.8) holds for $\widehat{\Omega}$ defined by (4.7), ensuring the compatibility of (4.8). With $M_n = (L^n)_{\geq k}$, $M' = \sum_i \phi_i \partial_x^{-1} \psi_i \, \partial_x^k$ and $\widehat{\Omega}_i = \widehat{\Omega}(\psi_i, \phi^{(k)})$ one obtains

$$\begin{aligned} \phi_{zt_n} &= \textstyle\sum_i [M_n \phi_i] \, \widehat{\Omega}_i \;+\; \sum_i \mathrm{res}(\phi_i \partial_x^{-1} \psi_i \partial_x^k M_n \partial_x^{-k} \phi^{(k)} \partial_x^{-1}) \\ &= \textstyle\sum_i [M_n \phi_i] \, \widehat{\Omega}_i \;+\; \mathrm{res}(M' M_n \partial_x^{-k} \phi^{(k)} \partial_x^{-1}) \,. \end{aligned}$$

Insertion of $M' M_n = M_{nz} - M'_{t_n} + M_n M'$ (from (A.12)) into the last equation

[1] It is useful to note that

$$\psi_{t_n} = -(\partial_x^{-k} (L^n)^*_{\geq k} \partial_x^k \psi)_0 = -\mathrm{res}(\partial_x^{-k} (L^n)^*_{\geq k} \partial_x^k \psi \partial_x^{-1}) \stackrel{(A.3)}{=} -\mathrm{res}(\partial_x^{-1} \psi (L^n)_{\geq k} \partial_x^{-k}) \,.$$

produces

$$\phi_{zt_n} = \sum_i [M_n \phi_i]\,\widehat{\Omega}_i \;+\; \mathrm{res}(M_{nz}\partial_x^{-k}\phi^{(k)}\partial_x^{-1}) \\ \;-\; \mathrm{res}(M'_{t_n}\partial_x^{-k}\phi^{(k)}\partial_x^{-1}) \;+\; \mathrm{res}(M_n M'\partial_x^{-k}\phi^{(k)}\partial_x^{-1})\,.$$

With (A2) the second term yields $[M_{nz}\phi]$. The third term vanishes since $M'_{t_n}\partial_x^{-k} \in g_{<0}$. The fourth term yields

$$\begin{aligned}\sum_i \mathrm{res}(M_n\phi_i\partial_x^{-1}\psi_i\phi^{(k)}\partial_x^{-1}) \;&=\; \sum_i \mathrm{res}(M_n\phi_i\partial_x^{-1}(\partial_x\widehat{\Omega}_i - \widehat{\Omega}_i\partial_x)\partial_x^{-1}) \\ &\overset{(A.2)}{=} \sum_i [M_n\phi_i\widehat{\Omega}_i] \;-\; \sum_i [M_n\phi_i]\,\widehat{\Omega}_i\,,\end{aligned}$$

whence $\phi_{zt_n} = [M_{nz}\phi] + \sum_i [M_n\phi_i\widehat{\Omega}_i] = [M_{nz}\phi + M_n\phi_z] = \phi_{t_nz}$. The compatibility of (4.5) is checked by a similar computation.

□

Proof of theorem 4.4: Using (4.9) one computes

$$\begin{aligned}&(M'_{z_2} - M''_{z_1} - [M'', M'])\partial_x^k = \\ &\quad \sum_i \Big(\phi_{iz_2} - \sum_j q_j\widehat{\Omega}(r_j,\phi_i)\Big)\partial_x^{-1}\psi_i + \sum_i \phi_i\partial_x^{-1}\Big(\psi_{iz_2} - (-1)^k \sum_j \Omega(\psi_i, q_j) r_j\Big) \\ &- \sum_j \Big(q_{jz_1} - \sum_i \phi_i\widehat{\Omega}(\psi_i, q_j)\Big)\partial_x^{-1} r_j + \sum_j q_j\partial_x^{-1}\Big(r_{jz_1} - (-1)^k \sum_i \Omega(r_j,\phi_i)\psi_i\Big)\,.\end{aligned}$$

□

Proof of theorem 4.6: a) From

$$\tilde{M}_n := (\tilde{L}^n)_{\geq 1} = (\phi^{-1}M_n\phi)_{\geq 1} = \phi^{-1}M_n\phi - \phi^{-1}[M_n\phi]$$

with $M_n := (L^n)_{\geq 0}$ one finds

$$\tilde{\phi}_{it_n} - [\tilde{M}_n\tilde{\phi}_i] = \phi^{-1}(\phi_{it_n} - [M_n\phi_i]) \;-\; \phi^{-1}(\phi_{t_n} - [M_n\phi])\phi^{-1}\phi_i = 0\,.$$

From the definition of $\Omega(\psi_i,\phi)$ and $\phi\tilde{M}_n = M_n\phi - [M_n\phi]$ one computes

$$\begin{aligned}
\tilde{\psi}_{it_n} &= -\mathrm{res}(\partial_x^{-1}\psi_i M_n\phi\,\partial_x^{-1}) \;=\; -\,\mathrm{res}(\partial_x^{-1}\psi_i\phi\tilde{M}_n\partial_x^{-1}) \\
&= \mathrm{res}(\partial_x^{-1}(\partial_x\tilde{\psi}_i - \tilde{\psi}_i\partial_x)\tilde{M}_n\partial_x^{-1}) \;=\; -\,\mathrm{res}(\partial_x^{-1}\tilde{\psi}_i\partial_x\tilde{M}_n\partial_x^{-1}) \\
&\overset{(A.3)}{=} -\mathrm{res}(\partial_x^{-1}\tilde{M}_n^*\partial_x\tilde{\psi}_i\partial_x^{-1}) \;\overset{(A.2)}{=}\; -\,[\partial_x^{-1}\tilde{M}_n^*\partial_x\tilde{\psi}_i]\,.
\end{aligned}$$

Further,

$$\begin{aligned}
\tilde{L}_z \;&=\; [\,-\phi^{-1}\phi_z\,,\,\tilde{L}\,] + \phi^{-1}L_z\phi \;=\; [\,\textstyle\sum_i(\phi^{-1}\phi_i\tilde{\psi}_i + \phi^{-1}\phi_i\partial_x^{-1}\psi_i\phi)\;,\;\tilde{L}\,] \\
&= [\,\textstyle\sum_i(\tilde{\phi}_i\tilde{\psi}_i - \tilde{\phi}_i\partial_x^{-1}\tilde{\psi}_{ix})\,,\;\tilde{L}\,] \;=\; [\,\textstyle\sum_i\tilde{\phi}_i\partial_x^{-1}\tilde{\psi}_i\partial_x\;,\;\tilde{L}\,]\,.
\end{aligned}$$

b) Since the first order coefficient of $(\bar{L}^n)_{\overline{\geq 1}} = (\tilde{L}^n)_{\geq 1}$ is given by $[(\bar{L}^n)_{\overline{\geq 1}}\,\bar{x}]$, one has $\bar{M}_n := (\bar{L}^n)_{\overline{\geq 2}} = \tilde{M}_n - [\tilde{M}_n\tilde{\phi}]\partial_{\bar{x}}$, whence

$$\bar{\phi}_{i\bar{t}_n} = \tilde{\phi}_{it_n} - \tilde{\phi}_{it_n}\bar{\phi}_{i\bar{x}} = [\tilde{M}_n\tilde{\phi}_i - [\tilde{M}_n\tilde{\phi}]\bar{\phi}_{i\bar{x}}] = [\bar{M}_n\bar{\phi}_i]\,.$$

According to (4.8) the potential $\widehat{\Omega}_i = \widehat{\Omega}(\tilde{\psi}_i,\tilde{\phi}_x) = \tilde{\psi}_i\tilde{\phi} - \Omega(\tilde{\psi}_{ix},\tilde{\phi}) = -\bar{\psi}_i$ satisfies $\widehat{\Omega}_{ix} = \tilde{\psi}_i\tilde{\phi}_x$ and $\widehat{\Omega}_{it_n} = \mathrm{res}(\partial_x^{-1}\tilde{\psi}_i\partial_x\tilde{M}_n\partial_x^{-1}\tilde{\phi}_x\partial_x^{-1})$, whence

$$\begin{aligned}
\bar{\psi}_{i\bar{t}_n} &= -\widehat{\Omega}_{it_n} + \widehat{\Omega}_{ix}\tilde{\phi}_x^{-1}\tilde{\phi}_{t_n} = -\mathrm{res}(\partial_x^{-1}\tilde{\psi}_i\partial_x\tilde{M}_n\partial_x^{-1}\tilde{\phi}_x\partial_x^{-1}) + \tilde{\psi}_i\tilde{\phi}_{t_n} \\
&\overset{(*)}{=} \mathrm{res}(\partial_x^{-1}\tilde{\psi}_{ix}\tilde{M}_n\partial_x^{-1}\tilde{\phi}_x\partial_x^{-1}) \overset{(**)}{=} \mathrm{res}(\partial_{\bar{x}}^{-1}\tilde{\phi}_x^{-1}\tilde{\psi}_{ix}\tilde{M}_n\partial_{\bar{x}}^{-2}\tilde{\phi}_x^{-1}) \\
&\overset{(***)}{=} \overline{\mathrm{res}}(\partial_{\bar{x}}^{-1}\tilde{\phi}_x^{-1}\tilde{\psi}_{ix}\bar{M}_n\partial_{\bar{x}}^{-2})\,.
\end{aligned}$$

We note $\tilde{\psi}_i\tilde{\phi}_{t_n} = \tilde{\psi}_i[\tilde{M}_n\tilde{\phi}] = \mathrm{res}(\tilde{\psi}_i\tilde{M}_n\partial_x^{-1}\tilde{\phi}_x\partial_x^{-1})$ in $(*)$ and $\partial_x^{-1}\tilde{\phi}_x = \partial_{\bar{x}}^{-1}$ in $(**)$. The residue $\overline{\mathrm{res}}$ related to the symbol $\partial_{\bar{x}}$ is given by $\overline{\mathrm{res}}(\bar{A}) = \mathrm{res}(A\tilde{\phi}_x^{-1})$ for all symbols $\bar{A} = A$. In $(***)$ one may replace $\tilde{M}_n$ by $\bar{M}_n$, since the difference $\tilde{M}_n - \bar{M}_n = [\tilde{M}_n\tilde{\phi}]\partial_{\bar{x}}$ does not contribute to the residue. Observing $\bar{\psi}_{i\bar{x}} = -\widehat{\Omega}(\tilde{\psi}_i,\tilde{\phi}_x)_x\tilde{\phi}_x^{-1} = -\tilde{\psi}_i$ one obtains

$$\bar{\psi}_{i\bar{t}_n} = \overline{\mathrm{res}}(\partial_{\bar{x}}^{-1}\tilde{\phi}_x^{-1}\tilde{\psi}_{ix}\bar{M}_n\partial_{\bar{x}}^{-2}) = -\overline{\mathrm{res}}(\partial_{\bar{x}}^{-1}\bar{\psi}_{i\bar{x}\bar{x}}\bar{M}_n\partial_{\bar{x}}^{-2}) = -[\partial_{\bar{x}}^{-2}\bar{M}_n\partial_{\bar{x}}^2\bar{\psi}_i]\,.$$

Finally, $\tilde{L}_z = [\partial_z, \tilde{L}] = [\partial_{\bar{z}} + \tilde{\phi}_z \partial_{\bar{x}}, \bar{L}] = \bar{L}_{\bar{z}} + [\tilde{\phi}_z \tilde{\phi}_x^{-1} \partial_x, \tilde{L}]$ yields

$$\begin{aligned} \bar{L}_{\bar{z}} &= \tilde{L}_z - [\tilde{\phi}_z \tilde{\phi}_x^{-1} \partial_x, \tilde{L}\,] = [\,\textstyle\sum_i (\tilde{\phi}_i \partial_x^{-1} \tilde{\psi}_i \partial_x - \tilde{\phi}_i \widehat{\Omega}_i \tilde{\phi}_x^{-1} \partial_x)\,,\ \tilde{L}\,] \\ &= [\,\textstyle\sum_i (-\bar{\phi}_i \partial_{\bar{x}}^{-1} \bar{\psi}_{i\bar{x}} \partial_{\bar{x}} + \bar{\phi}_i \bar{\psi}_i \partial_{\bar{x}}\,,\ \bar{L}\,] = [\,\textstyle\sum_i \bar{\phi}_i \partial_{\bar{x}}^{-1} \bar{\psi}_i \bar{\partial}_x^2\,,\ \bar{L}\,]\,. \end{aligned}$$

$\square$