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Hamiltonian structures of the Melnikov system and its reductions

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Abstract

The bi-Hamiltonian structure of an integrable dynamical system introduced by Melnikov is studied. This equation arises as a symmetry constraint of the KP hierarchy via squared eigenfunctions and can be understood as a Boussinesq–system with a source. The standard linear and quadratic Poisson brackets associated with the space of pseudo-differential symbols are used to derive two compatible Hamiltonian operators. Via reduction techniques a bi-Hamiltonian formulation for the Drinfeld-Sokolov system is derived.

1 Introduction

The interrelations between $1+1$–dimensional integrable systems and $1+0$–dimensional integrable systems are well known in soliton theory. Many finite dimensional integrable Hamiltonian systems can be obtained by reduction schemes involving pole expansions, stationary flows or reductions to pure soliton manifolds [1–5]. An early example for the latter scheme is provided by the restriction of the KdV flow to the pure multisoliton manifold where the KdV potential u is identified with the squared eigenfunctions ψ^2 [5]. This identification has been systematically developed to a nonlinearization procedure [6–10]. Since the quantity ψ^2 can be viewed as the gradient of the (conserved) spectral parameter the restriction has been generalized to the identification of an arbitrary KdV symmetry with $\psi\psi_x$. An important step in proving the integrability of those reduced systems was the discovery of the Lax pair

$$[\partial^2 + u, B + \psi\partial^{-1}\psi^*] = 0\,, \tag{1.1}$$

where B is a differential operator commuting with the Schrödinger operator up to terms of zero order. This has been generalized to n-th order operators [11–13].

It is also possible to reduce soliton equations in $2+1$ dimensions to simpler integrable equations in $1+1$ dimensions [14–19]. The KP equation as a two–dimensional generalization of the KdV is the most widely considered example. Here the quantity $(\psi\psi^*)_x$ with eigenfunctions ψ and adjoint eigenfunctions ψ^* plays the role of a symmetry generator which is identified with a KP symmetry. Through such symmetry constraints the whole KP hierarchy is restricted to hierarchies of commuting flows in $1+1$ dimensions. Instead of a differential operator l of order n a pseudo–differential operator

$$L = l + \psi\partial^{-1}\psi^* \tag{1.2}$$

with a so–called source term gives rise to a Lax formulation. Recently, starting from this observation the bi–Hamiltonian structure of the reduced hierarchies has been revealed. The first three cases, i.e. the order $n = 1, 2, 3$ have been considered in various papers [17–19]. They are leading to the AKNS, the Yajima–Oikawa and the Melnikov systems, respectively.

The aim of this paper is to derive the bi-Hamiltonian structures of the Melnikov system and the Drinfeld-Sokolov system which arises as a reduction. These results are derived from a more general background presented in [25], where a general construction of the bi-Hamiltonian structure associated with Lax operators of the form (1.2) is given. There it was found that the standard linear and quadratic Poisson brackets on the space of pseudo-differential symbols admit proper restrictions to operators with a pseudo-differential part of the special form indicated in (1.2). It turns out that because of the special parametrization by a pair of eigenfunctions and adjoint eigenfunctions the abstract quadratic bracket becomes non-local. A suitable representation of the Hamiltonian structures is given in [25], here we apply these results to the special case of third order operators and the associated Melnikov hierarchy. Although a straightforward application of the general theory, the required computations are cumbersome. The Hamiltonian operators derived from the general abstract brackets are given explicitly in Sections 2,3 and 4.

In Section 2 the results associated with the purely differential third order Lax operator $l = \partial^3 + p_1\partial + p_0$ are reviewed. The corresponding integrable nonlinear soliton equation is the Boussinesq system which may be regarded as a Melnikov system without source. The well known bi-Hamiltonian structure is given by the standard linear Lie-Poisson bracket and the quadratic Gelfand-Dikii bracket associated with differential operators.

In Section 3 a pseudo-differential term is added to the Lax operator which results in an additional source term in the Boussinesq system. This set of equations, together with the corresponding time evolutions of the (adjoint) eigenfunctions, represents the Melnikov system. We use the abstract brackets of [25] to derive two Hamiltonian operators for the corresponding hierarchy of equations.

In Section 4 the reduction of the Melnikov hierarchy to the Drinfeld-Sokolov system and its higher flows is considered. Here the first higher symmetry is interesting in its own right, since it represents the Kupershmidt equation coupled with a source term. The standard process of Dirac reduction yields a simple local Hamiltonian operator from the (non-local) second Hamiltonian structure of the Melnikov hierarchy. The first Poisson bracket cannot be reduced this way. Nevertheless a further (symplectic) Hamiltonian formulation of the Drinfeld-Sokolov system is obtained by generating a symplectic operator from the two Hamiltonian operators of the

Melnikov system. A simple pull-back of this structure through the reduction yields the desired result.

In Section 5 we mention that the systems considered here arise as symmetry reductions of the 2+1 dimensional Kadomtsev-Petviashvili (KP) equation with a symmetry generated by squared eigenfunctions. Conversely, the 1+1 dimensional hierarchies provide non-trivial factorizations of the KP, in that a simultaneous solution of two 1+1 dimensional flows satisfy the KP equation.

Finally, we remark that these studies can be generalized easily to the case of orders $n > 3$ for the Lax operator L.

2 The Boussinesq system

Let us consider the following differential operator of third order

$$l = \partial^3 + p_1\partial + p_0 \tag{2.1}$$

together with its fractional powers. This gives rise to the hierarchy of Lax equations

$$l_{t_q} = [(l^{\frac{q}{3}})_{\geq 0}, l]\,, \tag{2.2}$$

where $q = 1, 2, \ldots$. (If we have a pseudo–differential operator O then $O_{\geq 0}$ denotes its purely differential part and partial derivatives with respect to some variable are denoted by subscripts). The first nontrivial system obtained from (2.2) is the Boussinesq-system

$$p_{1,t_2} = -p_{1,xx} + 2p_{0,x}\,, \tag{2.3}$$

$$p_{0,t_2} = -\frac{2}{3}p_{1,xxx} - \frac{2}{3}p_1 p_{1,x} + p_{0,xx}\,. \tag{2.4}$$

It is well–known that the Lax equations (2.2) possess two Hamiltonian structures [20–22], namely a linear structure given by

$$l_{t_q} = \left[l, \frac{\delta H}{\delta l}\right]_{\geq 0}\,, \tag{2.5}$$

where $H = \frac{3}{q+3}\mathrm{res}\left(l^{\frac{q+3}{3}}\right)$, and a quadratic structure given by

$$l_{t_q} = \left(l\frac{\delta H}{\delta l}\right)_{\geq 0} l - l\left(\frac{\delta H}{\delta l}l\right)_{\geq 0}\,, \tag{2.6}$$

where $H = \frac{3}{q}\mathrm{res}\left(l^{\frac{q}{3}}\right)$. (As usual, we denote by $\mathrm{res}(O)$ the residue of a pseudo–differential operator O, i.e. its coefficient of the symbol ∂^{-1}).

By working out the Hamiltonian structures (2.5) and (2.6) we obtain Hamiltonian formulations for the systems of the Boussinesq hierarchy

$$\begin{pmatrix} p_1 \\ p_0 \end{pmatrix}_{t_q} = M_1 \bigtriangledown \left(\frac{3}{q+3}\mathrm{res}\left(l^{\frac{q+3}{3}}\right)\right)\,, \tag{2.7}$$

$$\begin{pmatrix} p_1 \\ p_0 \end{pmatrix}_{t_q} = M_2 \bigtriangledown \left(\frac{3}{q}\mathrm{res}\left(l^{\frac{q}{3}}\right)\right)\,, \tag{2.8}$$

with Hamiltonian operators

$$M_1 = \begin{pmatrix} 0 & 3\partial \\ 3\partial & 0 \end{pmatrix}, \tag{2.9}$$

and

$$M_2 = \begin{pmatrix} \chi_{11} & \chi_{12} \\ \chi_{21} & \chi_{22} \end{pmatrix}, \tag{2.10}$$

where $\chi_{ij} = -\chi_{ji}^*$ and

$$\begin{aligned} \chi_{11} &= -\frac{2}{3}\partial^5 - \frac{2}{3}\left(\partial^3 p_1 + p_1\partial^3\right) + \partial\left(p_{0,x} - \frac{1}{3}p_1^2\right) + \left(p_{0,x} - \frac{1}{3}p_1^2\right)\partial\,, \\ \chi_{12} &= \partial^4 + p_1\partial^2 + 2p_0\partial + \partial p_0\,, \\ \chi_{22} &= 2\partial^3 + p_1\partial + \partial p_1\,. \end{aligned}$$

Finally, we mention that the Boussinesq hierarchy can be described by the recursion operator $M_1^{-1}M_2$ [23], which links

$$\nabla\left(\mathrm{res}\left(l^{\frac{q+3}{3}}\right)\right) = \frac{q+3}{q} M_1^{-1} M_2 \nabla\left(\mathrm{res}\left(l^{\frac{q}{3}}\right)\right)\,. \tag{2.11}$$

3 The Melnikov system and its Hamiltonian formulations

We now add a source term to the differential operator l yielding the pseudo–differential operator

$$L = l + \psi\partial^{-1}\psi^* = \partial^3 + p_1\partial + p_0 + \psi\partial^{-1}\psi^*\,. \tag{3.1}$$

The Lax equations (2.2) is now considered with L instead of l, i.e. the dynamical equations

$$L_{t_q} = [(L^{\frac{q}{3}})_{\geq 0}, L] \tag{3.2}$$

are assumed. The first nontrivial system in this hierarchy reads

$$p_{1,t_2} = -p_{1,xx} + 2p_{0,x}, \tag{3.3}$$

$$p_{0,t_2} = -\frac{2}{3}p_{1,xxx} - \frac{2}{3}p_1 p_{1,x} + p_{0,xx} + 2\left(\psi\psi^*\right)_x\,, \tag{3.4}$$

$$\psi_{t_2} = \psi_{xx} + \frac{2}{3}p_1\psi, \tag{3.5}$$

$$\psi^*_{t_2} = -\psi^*_{xx} - \frac{2}{3}p_1\psi^*\,. \tag{3.6}$$

Note that the variable p_0 can be eliminated from eq. (3.3) so that (3.4) becomes the usual Boussinesq–equation with a source term. System (3.3–3.6) then represents the stationary version of a system which has been introduced by Melnikov [24] as a model for self–exciting solitons.

The Hamiltonian operators for (3.3)-(3.6) are derived from the brackets (2.5) and (2.6), which are valid expressions not only for differential operators. The abstract description of the wanted Poisson structures are indeed obtained by simply replacing the differential l by the pseudo-differential L. In [25] it was shown that the resulting brackets admit a proper restriction to operators in which the pseudo-differential part is given in the special form indicated in (3.1). The technical problem of finding a suitable representation of the dual elements of $\delta H/\delta l$ was resolved in [25] by identifying extended brackets on the space of operators supplemented by pairs of (adjoint) eigenfunctions. For operators of the form $L = l + \psi\partial^{-1}\psi^*$ the corresponding Hamiltonian formulations were found to be given by

$$\begin{aligned} l_{t_q} &= \left[l, \frac{\delta H}{\delta l}\right]_{\geq 0}, && (3.7)\\ \psi_{t_q} &= \frac{\delta H}{\delta \psi^*}, && (3.8)\\ \psi^*_{t_q} &= -\frac{\delta H}{\delta \psi}, && (3.9)\end{aligned}$$

and

$$\begin{aligned} l_{t_q} &= \left(l\frac{\delta H}{\delta l}\right)_{\geq 0} l - l\left(\frac{\delta H}{\delta l}l\right)_{\geq 0} + \left(l\frac{\delta H}{\delta l}\psi\partial^{-1}\psi^*\right)_{\geq 0} \\ &\quad - \left(\psi\partial^{-1}\psi^*\frac{\delta H}{\delta \psi}l\right)_{\geq 0} + \left(l\frac{\delta H}{\delta \psi^*}\partial^{-1}\psi^*\right)_{\geq 0} - \left(\psi\partial^{-1}\frac{\delta H}{\delta \psi}l\right)_{\geq 0}, && (3.10)\\ \psi_{t_q} &= \left(l\frac{\delta H}{\delta l}\psi\right)_0 - \psi D^{-1}\left(\psi\frac{\delta H}{\delta \psi}\right) + \psi D^{-1}\left(\psi^*\frac{\delta H}{\delta \psi^*}\right) + \left(l\frac{\delta H}{\delta \psi^*}\right)_0, && (3.11)\\ \psi^*_{t_q} &= -\left(l(\frac{\delta H}{\delta l})^*\psi^*\right)_0 + \psi^* D^{-1}\left(\psi\frac{\delta H}{\delta \psi}\right) - \left(l^*\frac{\delta H}{\delta \psi}\right)_0 \\ &\quad -\psi^* D^{-1}\left(\psi^*\frac{\delta H}{\delta \psi^*}\right), && (3.12)\end{aligned}$$

Here the gradient of a Hamiltonian functional $H(l, \psi, \psi^*)$ of a differential operator l is identified as a pseudo-differential symbol in the standard way. The subscript $_0$ denotes the zero-order of the operator and D^{-1} denotes the integral operation $D^{-1}f(x) = \frac{1}{2}(\int_{-\infty}^x f(\xi)d\xi - \int_x^\infty f(\xi)d\xi)$, which should be distinguished from the pseudo–differential symbol ∂^{-1}. For technical details of this notation we refer to [25].

For later convenience, in particular for the reduction of the Hamiltonian structures in Section 4, we reparametrize the operator (3.1) in the form

$$L = \partial^3 + 3u\partial + \frac{3}{2}u_x + v + \psi\partial^{-1}\psi^*, \quad (3.13)$$

with $p_1 = 3u$ und $p_0 = 3/2u_x + v$. In these coordinates the first three evolution equations in the Melnikov hierarchy are given by

$$\begin{aligned} u_{t_2} &= \frac{2}{3}v_x, && (3.14)\\ v_{t_2} &= -\frac{1}{2}u_{xxx} - 6uu_x + 2(\psi\psi^*)_x, && (3.15)\\ \psi_{t_2} &= \psi_{xx} + 2u\psi, && (3.16)\\ \psi^*_{t_2} &= -\psi^*_{xx} - 2u\psi^*, && (3.17)\end{aligned}$$

$$
\begin{aligned}
u_{t_3} &= (\psi\psi^*)_x \,, & (3.18)\\
v_{t_3} &= \frac{3}{2}\left(\psi_{xx}\psi^* - \psi\psi^*_{xx}\right)\,, & (3.19)\\
\psi_{t_3} &= \psi_{xxx} + 3u\psi_x + \frac{3}{2}u_x\psi + v\psi\,, & (3.20)\\
\psi^*_{t_3} &= \psi^*_{xxx} + 3u\psi^*_x + \frac{3}{2}u_x\psi^* - v\psi^*\,, & (3.21)
\end{aligned}
$$

$$
\begin{aligned}
u_{t_4} &= \frac{2}{9}v_{xxx} + \frac{2}{3}\left(\psi_{xx}\psi^* - \psi\psi^*_{xx}\right) + \frac{4}{3}(uv)_x\,, & (3.22)\\
v_{t_4} &= \left(-\frac{1}{6}u_{xxxx} - 3uu_{xx} - \frac{3}{2}u_x^2 - 4u^3\right)_x + 4(u\psi\psi^*)_x + \frac{2}{3}(v^2)_x \\
&\quad -\frac{1}{3}(\psi\psi^*)_{xxx} + 2(\psi_{xx}\psi^* + \psi\psi^*_{xx})_x\,, & (3.23)\\
\psi_{t_4} &= \psi_{xxxx} + 4u\psi_{xx} + \left(4u_x + \frac{4}{3}v\right)\psi_x \\
&\quad + \left(\frac{5}{3}u_{xx} + 2u^2 + \frac{2}{3}v_x + \frac{4}{3}\psi\psi^*\right)\psi\,, & (3.24)\\
\psi^*_{t_4} &= -\psi^*_{xxxx} - 4u\psi^*_{xx} - \left(4u_x - \frac{4}{3}v\right)\psi^*_x \\
&\quad - \left(\frac{5}{3}u_{xx} + 2u^2 - \frac{2}{3}v_x + \frac{4}{3}\psi\psi^*\right)\psi^*\,. & (3.25)
\end{aligned}
$$

We observe that (3.18)–(3.21) represents a generalisation of the Drinfeld-Sokolov system.

In order to derive the Hamiltonian operators for these equations from the abstract formulations (3.7)–(3.9) and (3.10)–(3.12) one has to identify the pseudo-differential operator representing the gradient of a Hamiltonian $H(u, v, \psi, \psi^*)$. One finds

$$
\frac{\delta H}{\delta l} = \frac{3}{2}\partial^{-1}\frac{\delta H}{\delta v} - \frac{1}{2}\partial^{-2}\frac{\delta H}{\delta v}\partial + \frac{1}{3}\partial^{-2}\frac{\delta H}{\delta u} \tag{3.26}
$$

where $\delta H/\delta u$ and $\delta H/\delta v$ are the usual variational derivatives of H with respect to u and v (i.e. they are functions of the space variable x). Indeed, with $l = \partial^3 + 3u\partial + 3/2u_x + v$ one verifies

$$
\mathrm{res}\left(\frac{dl}{dt}\frac{\delta H}{\delta l}\right) = \frac{du}{dt}\frac{\delta H}{\delta u} + \frac{dv}{dt}\frac{\delta H}{\delta v} \tag{3.27}
$$

modulo perfect x-derivatives, so that the duality structure given by the operator residue corresponds to the usual duality of variational calculus. Inserting (3.13) into (3.7)–(3.9) and (3.10)–(3.12) one finds the Hamiltonian operators

$$
\tilde{M}_1 = \begin{pmatrix} 0 & \partial & 0 & 0 \\ \partial & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{3.28}
$$

and

$$
\tilde{M}_2 = (\tilde{\chi}_{i,j})_{i,j=1,\ldots,4}\,, \tag{3.29}
$$

where $\tilde{\chi}_{ij} = -\tilde{\chi}^*_{ji}$ and

$$
\begin{aligned}
\tilde{\chi}_{11} &= \frac{2}{9}\partial^3 + \frac{1}{3}\left(\partial u + u\partial\right) , \\
\tilde{\chi}_{12} &= v\partial + \frac{2}{3}v_x , \\
\tilde{\chi}_{13} &= \frac{1}{3}\left(\partial\psi + \psi\partial\right) , \\
\tilde{\chi}_{14} &= \frac{1}{3}\left(\partial\psi^* + \psi^*\partial\right) , \\
\tilde{\chi}_{22} &= -\frac{1}{6}\partial^5 - \frac{5}{4}\left(\partial^3 u + u\partial^3\right) + \frac{3}{4}\left(\partial^2 u_x - u_x\partial^2\right) \\
&\quad -3\left(\partial u^2 + u^2\partial\right) + 2\left(\psi\partial\psi^* + \psi^*\partial\psi\right) , \\
\tilde{\chi}_{23} &= -\frac{5}{4}\psi\partial^2 - \frac{5}{12}\partial^2\psi + \frac{1}{4}\psi_{xx} - 2u\psi , \\
\tilde{\chi}_{24} &= \frac{5}{4}\psi^*\partial^2 + \frac{5}{12}\partial^2\psi^* - \frac{1}{4}\psi^*_{xx} + 2u\psi^* , \\
\tilde{\chi}_{33} &= -\frac{4}{3}\psi D^{-1}\psi , \\
\tilde{\chi}_{34} &= \partial^3 + \frac{3}{2}\left(u\partial + \partial u\right) + v + \frac{4}{3}\psi\partial^{-1}\psi^* , \\
\tilde{\chi}_{44} &= -\frac{4}{3}\psi^* D^{-1}\psi^* .
\end{aligned}
$$

By using a different approach starting from the linear problems these structures were found in [26]. The Melnikov hierarchy is given by

$$
\begin{pmatrix} u \\ v \\ \psi \\ \psi^* \end{pmatrix}_{t_q} = \tilde{M}_1 \bigtriangledown \left(\frac{3}{q+3}\mathrm{res}\left(L^{\frac{q+3}{3}}\right)\right) = \tilde{M}_2 \bigtriangledown \left(\frac{3}{q}\mathrm{res}\left(L^{\frac{q}{3}}\right)\right) , \tag{3.30}
$$

where the first few Hamiltonians are computed as

$$
\mathrm{res}\left(L^{\frac{1}{3}}\right) = u , \tag{3.31}
$$

$$
\mathrm{res}\left(L^{\frac{2}{3}}\right) = \frac{2}{3}v , \tag{3.32}
$$

$$
\mathrm{res}\left(L^{\frac{3}{3}}\right) = \psi\psi^* , \tag{3.33}
$$

$$
\mathrm{res}\left(L^{\frac{4}{3}}\right) = \frac{4}{3}uv + \frac{2}{9}v_{xx} + \frac{2}{3}\left(\psi_x\psi^* - \psi\psi^*_x\right) , \tag{3.34}
$$

$$
\begin{aligned}
\mathrm{res}\left(L^{\frac{5}{3}}\right) &= -\frac{1}{9}u_{xxxx} - \frac{5}{3}uu_{xx} - \frac{5}{4}u_x^2 - \frac{5}{3}u^3 + \frac{5}{9}v^2 + \frac{10}{3}u\psi\psi^* \\
&\quad + \frac{10}{9}\left(\psi_{xx}\psi^* + \psi\psi^*_{xx}\right) + \frac{5}{9}\psi_x\psi^*_x ,
\end{aligned} \tag{3.35}
$$

$$
\mathrm{res}\left(L^{\frac{6}{3}}\right) = 3u\left(\psi_x\psi^* - \psi\psi^*_x\right) + 2v\psi\psi^* + \left(\psi_{xxx}\psi^* - \psi\psi^*_{xxx}\right) . \tag{3.36}
$$

The recursive relation between the Melnikov flows is given by

$$\nabla\left(\mathrm{res}\left(L^{\frac{q+3}{3}}\right)\right)=\frac{q+3}{q}\tilde{M_1}^{-1}\tilde{M_2}\,\nabla\left(\mathrm{res}\left(L^{\frac{q}{3}}\right)\right)\,. \tag{3.37}$$

4 The Drinfeld–Sokolov system

The Melnikov hierarchy possesses a nontrivial reduction

$$v=0\,,\quad \psi=\psi^*\,, \tag{4.1}$$

for which the Lax operator (3.13) reduces to the skew adjoint operator

$$L_r=\partial^3+3u\partial+\frac{3}{2}u_x+\psi\partial^{-1}\psi\,. \tag{4.2}$$

We remark that each second equation (3.2) with odd q survives this reduction. The reduction of the Melnikov system yields the Drinfeld-Sokolov system

$$u_{t_3} = (\psi^2)_x\,, \tag{4.3}$$

$$\psi_{t_3} = \psi_{xxx}+3u\psi_x+\frac{3}{2}u_x\psi\,, \tag{4.4}$$

Its first higher flow is given by

$$\begin{aligned} u_{t_5} &= -\frac{1}{9}\left(u_{xxxxx}+15uu_{xxx}+\frac{75}{2}u_xu_{xx}+45u^2u_x\right)\\ &\quad+\frac{10}{3}(u\psi^2)_x+\frac{20}{9}\psi\psi_{xxx}+\frac{10}{3}\psi_x\psi_{xx}\,, \end{aligned}\tag{4.5}$$

$$\begin{aligned} \psi_{t_5} &= \psi_{5x}+5u\psi_{xxx}+\frac{15}{2}u_x\psi_{xx}+\left(5u^2+\frac{35}{6}u_{xx}+\frac{10}{3}\psi^2\right)\psi_x\\ &\quad+\left(\frac{5}{3}u_{xxx}+5uu_x\right)\psi\,. \end{aligned}\tag{4.6}$$

which represents the Kupershmidt equation [27] coupled with a source term.

The Hamiltonian formulation for this reduced hierarchy can now be found from the results for the Melnikov flows via standard techniques. The Hamiltonian functions are obtained by a simple restriction of (3.31)–(3.36) with the constraints (4.1), i.e.

$$\mathrm{res}\left(L_r^{\frac{1}{3}}\right) = u\,, \tag{4.7}$$

$$\mathrm{res}\left(L_r^{\frac{3}{3}}\right) = \psi^2\,, \tag{4.8}$$

$$\begin{aligned} \mathrm{res}\left(L_r^{\frac{5}{3}}\right) &= -\frac{1}{9}u_{xxxx}-\frac{5}{3}uu_{xx}-\frac{5}{4}u_x^2-\frac{5}{3}u^3\\ &\quad+\frac{10}{3}u\psi^2+\frac{20}{9}\psi_{xx}\psi+\frac{5}{9}\psi_x^2\,, \end{aligned}\tag{4.9}$$

We note that $L_r^{1/3}$ is skew-symmetric, so that the even oder powers $L_r^{2n/3}$ are symmetric operators. Observing $\mathrm{res}(O)=-\mathrm{res}(O^*)$ for any pseudo-differential operator O one find $\mathrm{res}(L_r^{2n/3})=$

0. This corresponds to the fact that in (3.2) the constraints (4.1) are violated if q is even, so that only the odd flows reduce to the Melnikov hierarchy.

The Hamiltonian operators (3.28) and (3.29) do not allow a proper restriction to the constrained manifold (4.1). Hence Dirac reduction is invoked to obtain results for the Drinfeld-Sokolov hierarchy. For a description of this process in suitable notation we refer to [28]. However, this scheme only works for the second structure $\tilde{M}_2$, where the Dirac reduced operator is computed by a straighforward but lenghty calculation as

$$\tilde{M}_{2r} = \left(\tilde{\chi}_{r,ij}\right)_{i,j=1,\ldots,4} , \tag{4.10}$$

where $\tilde{\chi}_{r,ij} = -\left(\tilde{\chi}_{r,ji}\right)^*$ and

$$\begin{aligned} \tilde{\chi}_{r,11} &= \frac{2}{9}\partial^3 + \frac{1}{3}\left(u\partial + \partial u\right) , \\ \tilde{\chi}_{r,12} &= \frac{1}{3}\left(\partial\psi + \psi\partial\right) , \\ \tilde{\chi}_{r,22} &= \frac{1}{2}\partial^3 + \frac{3}{4}\left(u\partial + \partial u\right) . \end{aligned}$$

Unfortunately the first Hamiltonian operator (3.28) can not be reduced this way, because certain matrix entries vanish.

In order to obtain a second Hamiltonian formulation for the reduced system we first consider the operator

$$\tilde{J} = \tilde{M}_1^{-1}\tilde{M}_2\tilde{M}_1^{-1} = \begin{pmatrix} D^{-1}\tilde{\chi}_{22}D^{-1} & D^{-1}\tilde{\chi}_{21}D^{-1} & D^{-1}\tilde{\chi}_{24} & -D^{-1}\tilde{\chi}_{23} \\ D^{-1}\tilde{\chi}_{12}D^{-1} & D^{-1}\tilde{\chi}_{11}D^{-1} & D^{-1}\tilde{\chi}_{14} & -D^{-1}\tilde{\chi}_{13} \\ \tilde{\chi}_{42}D^{-1} & \tilde{\chi}_{41}D^{-1} & -\tilde{\chi}_{44} & \tilde{\chi}_{43} \\ -\tilde{\chi}_{32}D^{-1} & -\tilde{\chi}_{31}D^{-1} & \tilde{\chi}_{34} & -\tilde{\chi}_{33} \end{pmatrix} . \tag{4.11}$$

As $\tilde{M}_1$ and $\tilde{M}_2$ are compatible Hamiltonian operators, the operator $\tilde{J}$ is symplectic. The pull–back of the operator $\tilde{J}$ to the constrained manifold given by (4.1) is described by the matrix

$$\tilde{J}_r = \left(\eta_{ij}\right)_{i,j=1,2} , \tag{4.12}$$

where $\eta_{ij} = -\left(\eta_{ji}\right)^*$ and

$$\begin{aligned} \eta_{11} &= D^{-1}\left(-\frac{1}{6}\partial^5 - \frac{5}{4}\left(\partial^3 u + u\partial^3\right) + \frac{3}{4}\left(\partial^2 u_x - u_x\partial^2\right)\right. \\ &\qquad \left. -3\left(\partial u^2 + u^2\partial\right) + 4\psi\partial\psi\right)D^{-1} , \\ \eta_{12} &= 2D^{-1}\left(\frac{5}{4}\psi\partial^2 + \frac{5}{12}\partial^2\psi - \frac{1}{4}\psi_{xx} + 2u\psi\right) , \\ \eta_{22} &= 4\left(\frac{1}{2}\partial_x^3 + \frac{3}{4}\left(u\partial + \partial u\right) + \frac{4}{3}\psi D^{-1}\psi\right) . \end{aligned}$$

Thus we obtain the following two Hamiltonian formulations of the reduced system

$$\begin{pmatrix} u \\ \psi \end{pmatrix}_{t_{2n+1}} = \tilde{M}_{2,r} \bigtriangledown \left(\frac{3}{2n+1}\mathrm{res}\left(L_r^{\frac{2n+1}{3}}\right)\right) , \tag{4.13}$$

and

$$\tilde{J}_r \begin{pmatrix} u \\ \psi \end{pmatrix}_{t_{2n+1}} = \nabla \left(\frac{3}{2n+7} \mathrm{res} \left(L_r^{\frac{2n+7}{3}} \right) \right) , \tag{4.14}$$

while its Lax representation becomes

$$L_{r,t_{2n+1}} = \left[\left(L_r^{\frac{2n+1}{3}} \right)_{\geq 0} , L_r \right] . \tag{4.15}$$

It has been checked by symbolic computations that $\tilde{M}_r \tilde{J}_r$ provides a hereditary recursion operator. Hence, further Hamiltonian structures of the Drinfeld-Sokolov may be constructed by applying this recursion operator to $\tilde{M}_r$.

5 A symmetry constraint

In this section we regard the Melnikov system as a symmetry constraint of the KP hierarchy. The pseudo-differential operator underlying the KP hierarchy is usually written in the form

$$L_{KP} = \partial + u_2 \partial^{-1} + u_3 \partial^{-2} + u_4 \partial^{-3} + \dots \tag{5.1}$$

and the nonlinear evolution equations for the coefficients $u_2, u_3, \dots$... are obtained from the Lax equations

$$L_{KP,t_n} = [(L_{KP}^n)_{\geq 0}, L_{KP}] . \tag{5.2}$$

with $n = 1, 2, 3, \dots$. In particular, the field $u = u_2$ satisfies the KP equation

$$4u_{tx} = (u_{xxx} + 6uu_x)_x + 3u_{yy} \tag{5.3}$$

with $x = t_1$, $y = t_2$ and $t = t_3$. The Melnikov hierarchy is obtained by imposing the constraint

$$(L_{KP})^3 = L = \partial^3 + 3u\partial + \frac{3}{2}u_x + v + \psi \partial^{-1} \psi^* \tag{5.4}$$

which converts (5.2) into (3.2) with $q = n/3$ and $t_q = t_n$. Via this constraint all variables $u_2, u_3, \dots$ are expressed in terms of the fields u, v, ψ and ψ^* of the Melnikov system, one finds

$$L_{KP} = \partial + u\partial^{-1} + \left(-\frac{1}{2}u_x + \frac{1}{3}v \right) \partial^{-2} + \left(\frac{1}{6}u_{xx} - u^2 - \frac{1}{3}v_x + \frac{1}{3}\psi\psi^* \right) \partial^{-3} + \dots \tag{5.5}$$

from which the relations $u_2 = u$, $u_3 = -u_x/2 + v$ etc. are obtained.

The evolution equations (5.2)/(3.2) imply that ψ and ψ^* evolve according to

$$\psi_{t_n} = (L^{\frac{n}{3}})_+ \psi , \tag{5.6}$$

$$\psi^*_{t_n} = -(L^{\frac{n}{3}*})_+ \psi^* , \tag{5.7}$$

so that these fields may be regarded as (adjoint) eigenfunctions of the KP hierarchy. It is easy to see that the KP flows (5.2) with $n = 3$ imply

$$u_{t_3} = (\psi\psi^*)_x . \tag{5.8}$$

It is known that the derivative of squared eigenfunctions may be considered as a symmetry of the KP hierarchy. In this sense the Melnikov system is a symmetry reduction of the KP hierarchy via (5.6)–(5.8).

One interesting implication of this connection between the Melnikov system and the KP hierarchy is that solutions of this 1+1 dimensional hierarchy give rise to solutions of the 2+1 dimensional KP equation (5.3) and its higher flows. Solving simultaneously the first two flows (3.14)–(3.17) and (3.18)–(3.21) one finds a solution u of (5.3).

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