to appear in Physica A (1996)

A 3×3 matrix spectral problem for AKNS hierarchy and its binary Nonlinearization

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Abstract

A three-by-three matrix spectral problem for AKNS soliton hierarchy is proposed and the corresponding Bargmann symmetry constraint involved in Lax pairs and adjoint Lax pairs is discussed. The resulting nonlinearized Lax systems possess classical Hamiltonian structures, in which the nonlinearized spatial system is intimately related to stationary AKNS flows. These nonlinearized Lax systems also lead to a sort of involutive solutions to each AKNS soliton equation.

1 Introduction

Symmetry constraints have aroused an increasing interest in recent few years due to the important roles they play in soliton theory. Such a kind of very successful symmetry constraint method is the nonlinearization technique for Lax pairs of soliton hierarchies, including mono-nonlinearization proposed by Cao and Geng [7] [8] and further binary nonlinearization [18] [17] [14].

In general, one considers the complicated nonlinear problems to be solved in such a way to break nonlinear problems into several linear or smaller ones and then to solve these resulting problems. It is following this idea that one has introduced the method of Lax pair to study nonlinear soliton equations. The Lax pairs are always linear with respect to their eigenfunctions. Nevertheless, the nonlinearization technique puts this original object, the Lax pair, into a nonlinear and more complicated object, the nonlinearized Lax system. It seems to be not reasonable enough, but in fact, it provides an effective way, different from the usual one, to solve soliton equations. The main reason why the nonlinearization technique takes effect is that kind of specific symmetry constraints expressed through the variational derivative of the potential.

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A similar symmetry constraint procedure for bi-Hamiltonian soliton hierarchies is presented by Antonowicz and Wojciechowski et al [2] [3] [22] and bi-Hamiltonian structures for the resulting classical integrable systems can also be worked out through a Miura map [3] [6]. A connection between these systems and stationary flows [5] is also given by Tondo [24] for the case of KdV hierarchy. Because stationary flows may be interpreted as finite dimensional Hamiltonian systems [5] based upon the so-called Jacobi-Ostrogradsky coordinates [16], a natural generalization of nonlinearization technique to higher order symmetry constraints is made by Zeng [27] [28] for the KdV and Kaup-Newell hierarchies etc. There have also been some algebraic geometric tricks, proposed by Flaschka et al [11] [1] [23], to deal with similar nonlinearized Lax pairs called Neumann systems.

The study of the nonlinearization theory leads to a large class of interesting finite dimensional Liouville integrable Hamiltonian systems which are connected with soliton hierarchies (for example, see [8] [15]). However in the literature, most results are presented for the cases of 2×2 matrix spectral problems. The present paper is devoted to the symmetry constraints in binary nonlinearization for a case of 3×3 matrix spectral problems. We successfully propose a 3×3 matrix spectral problem for AKNS soliton hierarchy, motivated by a representation of 3×3 matrices for the Lie algebra sl(2). Then in Section 3, we consider the Bargmann symmetry constraint for the proposed new Lax pairs and adjoint Lax pairs of AKNS soliton hierarchy. In Section 4, we analyze the nonlinearized Lax systems, especially the nonlinearized temporal systems, and establish a sort of involutive solutions to AKNS soliton equations. Finally in Section 5, some remarks are given.

2 New Lax pairs for AKNS equations

We introduce a three-by-three matrix spectral problem

$$\phi_x=\left(\begin{array}{c}\phi_1\\ \phi_2\\ \phi_3\end{array}\right)_x=U(u,\lambda)\left(\begin{array}{c}\phi_1\\ \phi_2\\ \phi_3\end{array}\right)=\left(\begin{array}{ccc}-2\lambda & \sqrt{2}q & 0\\ \sqrt{2}r & 0 & \sqrt{2}q\\ 0 & \sqrt{2}r & 2\lambda\end{array}\right)\left(\begin{array}{c}\phi_1\\ \phi_2\\ \phi_3\end{array}\right), \qquad (2.1)$$

where the potential $u=(q,r)^T$. Its adjoint spectral problem reads as

$$\psi_x=\left(\begin{array}{c}\psi_1\\ \psi_2\\ \psi_3\end{array}\right)_x=-U^T(u,\lambda)\left(\begin{array}{c}\psi_1\\ \psi_2\\ \psi_3\end{array}\right)=\left(\begin{array}{ccc}2\lambda & -\sqrt{2}r & 0\\ -\sqrt{2}q & 0 & -\sqrt{2}r\\ 0 & -\sqrt{2}q & -2\lambda\end{array}\right)\left(\begin{array}{c}\psi_1\\ \psi_2\\ \psi_3\end{array}\right). \quad (2.2)$$

Here T means the transposition of the matrix. Our purpose is to generate AKNS hierarchy of soliton equations from the above specific spectral problem (2.1). To this end, we first solve the adjoint representation equation $V_x=[U,V]$. Take

$$V=\left(\begin{array}{ccc}2a & \sqrt{2}b & 0\\ \sqrt{2}c & 0 & \sqrt{2}b\\ 0 & \sqrt{2}c & -2a\end{array}\right)=\sum_{i=0}^{\infty}\left(\begin{array}{ccc}2a_i & \sqrt{2}b_i & 0\\ \sqrt{2}c_i & 0 & \sqrt{2}b_i\\ 0 & \sqrt{2}c_i & -2a_i\end{array}\right)\lambda^{-i} \qquad (2.3)$$

and then we have

$$[U,V]=\left(\begin{array}{ccc} 2(qc-rb) & -2\sqrt{2}(\lambda b+qa) & 0 \\ 2\sqrt{2}(ra+\lambda c) & 0 & -2\sqrt{2}(\lambda b+qa) \\ 0 & 2\sqrt{2}(ra+\lambda c) & -2(qc-rb) \end{array}\right).$$

Therefore we easily find that the adjoint representation equation $V_x=[U,V]$ becomes

$$a_x=qc-rb,\ b_x=-2\lambda b-2qa,\ \ c_x=2\lambda c+2ra,$$

which is equivalent to

$$a_{ix}=qc_i-rb_i,\ b_{ix}=-2b_{i+1}-2qa_i,\ c_{ix}=2c_{i+1}+2ra_i,\ i\ge 0. \tag{2.4}$$

We fix the initial values

$$a_0=-1,\ b_0=c_0=0 \tag{2.5}$$

and require that

$$a_i|_{u=0}=b_i|_{u=0}=c_i|_{u=0}=0,\ i\ge 1, \tag{2.6}$$

which equivalently select constants of integration to be zero. On the other hand, the above equality (2.4) gives rise to the recursion relation for determining a_i,b_i,c_i:

$$\left\{\begin{array}{l} a_{i+1}=\frac{1}{2}\partial^{-1}(qc_{ix}+rb_{ix}), \\ b_{i+1}=-\frac{1}{2}b_{ix}-qa_i, \\ c_{i+1}=\frac{1}{2}c_{ix}-ra_i, \end{array}\right. \quad i\ge 0. \tag{2.7}$$

This recursion relation uniquely determines infinitely many sets of polynomials $a_i,b_i,c_i,\ i\ge 1$, in $u,u_x,\cdots$ under the requirement (2.6). The first two sets are as follows

$$a_1=0,\ b_1=q,\ c_1=r;\ a_2=\frac{1}{2}(qr),\ b_2=-\frac{1}{2}q_x,\ c_2=\frac{1}{2}r_x.$$

In addition, we have

$$a^2+bc=(\sum_{i=0}^{\infty}a_i\lambda^{-i})^2+(\sum_{i=0}^{\infty}b_i\lambda^{-i})(\sum_{i=0}^{\infty}c_i\lambda^{-i})=1,$$

because $(a^2+bc)_x=\frac{1}{8}\mathrm{tr}(V^2)_x=\frac{1}{8}\mathrm{tr}[U,V^2]=0$ and $(a^2+bc)|_{u=0}=1$. It follows that $a_i,b_i,c_i,\ i\ge 1$, are local.

A direct computation may show that the compatibility conditions of the Lax pairs

$$\phi_x=U\phi,\ \phi_{t_n}=V^{(n)}\phi,\ V^{(n)}=V^{(n)}(u,\lambda)=(\lambda^nV)_+,\ n\ge 0, \tag{2.8}$$

or the adjoint Lax pairs

$$\psi_x=-U^T\phi,\ \psi_{t_n}=-(V^{(n)})^T\psi,\ n\ge 0, \tag{2.9}$$

where the symbol + denotes the choice of non-negative power of λ, engenders a hierarchy of AKNS soliton equations

$$u_{t_n} = \left(\begin{array}{c} q \\ r \end{array}\right)_{t_n} = K_n = \left(\begin{array}{c} -2b_{n+1} \\ 2c_{n+1} \end{array}\right) = JL^n \left(\begin{array}{c} r \\ q \end{array}\right), \ n \geq 0, \tag{2.10}$$

where the Hamiltonian operator J and the recursion operator L read as

$$J = \left(\begin{array}{cc} 0 & -2 \\ 2 & 0 \end{array}\right), \ L = \left(\begin{array}{cc} \frac{1}{2}\partial - r\partial^{-1}q & r\partial^{-1}r \\ -q\partial^{-1}q & -\frac{1}{2}\partial + q\partial^{-1}r \end{array}\right). \tag{2.11}$$

This AKNS hierarchy is exactly the same as one in Ref. [18], which also shows that the same soliton hierarchy may possess different Lax pairs, even different order spectral matrices. Here the operator L^* is a hereditary operator [13], and J and JL constitute a Hamiltonian pair.

Finally, we would like to elucidate the other two properties on AKNS hierarchy (2.10). First by Corollary 2.1 of Ref. [18], we can obtain

$$V_{t_n} = [V^{(n)}, V], \ n \geq 0 \tag{2.12}$$

when $u_{t_n} = K_n$, i.e. $U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0, \ n \geq 0$. Second, we can get the Hamiltonian structure of AKNS hierarchy

$$u_{t_n} = K_n = J\left(\begin{array}{c} c_{n+1} \\ b_{n+1} \end{array}\right) = J\frac{\delta H_n}{\delta u}, \ H_n = \frac{2}{n+1}a_{n+2}, \ n \geq 0, \tag{2.13}$$

by applying the trace identity [25] [26].

3 Binary nonlinearization related to new spectral problem

In order to impose the Bargmann symmetry constraint in binary nonlinearization, we first need to compute the variational derivative of the spectral parameter λ with to the potential u, which is shown in the following Lemma [12][18].

Lemma 3.1 *Let $U(u,\lambda)$ be a matrix of order s depending on u, u_x, $\cdots$ and a parameter λ. Suppose that $\phi = (\phi_1, \phi_2, \cdots, \phi_s)^T$, $\psi = (\psi_1, \psi_2, \cdots, \psi_s)^T$ satisfy the spectral problem and the adjoint spectral problem*

$$\phi_x = U(u,\lambda)\phi, \ \psi_x = -U^T(u,\lambda)\psi,$$

and set the matrix $\bar{V} = \phi\psi^T = (\phi_k\psi_l)_{s\times s}$, then we have the following two results:

(i) the variational derivative of the spectral parameter λ with respect to the potential u may be expressed by

$$\frac{\delta\lambda}{\delta u} = \frac{tr(\bar{V}\frac{\partial U}{\partial u})}{-\int_{-\infty}^{\infty} tr(\bar{V}\frac{\partial U}{\partial \lambda})dx}, \tag{3.1}$$

(ii) the matrix $\bar V$ is a solution to the adjoint representation equation $V_x = [U, V]$, i.e. $\bar V_x = [U, \bar V]$.

Following (3.1), we have the variational derivative of the spectral parameter for the spectral problem (2.1) and the adjoint spectral problem (2.2)

$$\frac{\delta\lambda}{\delta q} = \frac{\sqrt{2}}{E}(\phi_2\psi_1 + \phi_3\psi_2),\ \frac{\delta\lambda}{\delta r} = \frac{\sqrt{2}}{E}(\phi_1\psi_2 + \phi_2\psi_3), \tag{3.2}$$

where $E = 2\int_{-\infty}^{\infty}(\phi_1\psi_1 - \phi_3\psi_3)dx$.

Let us introduce N ($N \geq 1$) distinct eigenvalues λ_j, $1 \leq j \leq N$, and denote by

$$\phi^{(j)} = (\phi_{1j}, \phi_{2j}, \phi_{3j})^T,\ \psi^{(j)} = (\psi_{1j}, \psi_{2j}, \psi_{3j})^T,\ 1 \leq j \leq N,$$

the eigenfunctions of (2.8) and the adjoint eigenfunctions of (2.9), i.e.

$$\phi^{(j)}_x = U(u, \lambda_j)\phi^{(j)},\ \psi^{(j)}_x = -U^T(u, \lambda_j)\psi^{(j)},\ 1 \leq j \leq N, \tag{3.3}$$

$$\phi^{(j)}_{t_n} = V^{(n)}(u, \lambda_j)\phi^{(j)},\ \psi^{(j)}_{t_n} = -(V^{(n)})^T(u, \lambda_j)\psi^{(j)},\ 1 \leq j \leq N. \tag{3.4}$$

Now we make the Bargmann symmetry constraint

$$K_0 = J\frac{\delta H_0}{\delta u} = J\sum_{j=1}^N \mu_j E_j \frac{\delta \lambda_j}{\delta u}, \tag{3.5}$$

where $E_j = 2\int_{-\infty}^{\infty}(\phi_{1j}\psi_{1j} - \phi_{3j}\psi_{3j})dx$, $1 \leq j \leq N$, and μ_j, $1 \leq j \leq N$, are any nonzero constants. By (3.2), this symmetry constraint becomes

$$K_0 = J\sum_{j=1}^N \mu_j \begin{pmatrix} \sqrt{2}(\phi_{2j}\psi_{1j} + \phi_{3j}\psi_{2j}) \\ \sqrt{2}(\phi_{1j}\psi_{2j} + \phi_{2j}\psi_{3j}) \end{pmatrix},$$

from which we get the following explicit expression for the potential u

$$u = f(P_1, P_2, P_3; Q_1, Q_2, Q_3) = \sqrt{2}\begin{pmatrix} < P_1, BQ_2 > + < P_2, BQ_3 > \\ < P_2, BQ_1 > + < P_3, BQ_2 > \end{pmatrix}. \tag{3.6}$$

Here and hereafter, $< \cdot, \cdot >$ denotes the standard inner product of $\mathbb{R}^N$ and

$$B = \mathrm{diag}(\mu_1, \cdots, \mu_N),\ \begin{pmatrix} P_i \\ Q_i \end{pmatrix} = \begin{pmatrix} (\phi_{i1}, \phi_{i2}, \cdots, \phi_{iN})^T \\ (\psi_{i1}, \psi_{i2}, \cdots, \psi_{iN})^T \end{pmatrix},\ i = 1, 2, 3. \tag{3.7}$$

The substitution of (3.6) into the spatial system (3.3) and the temporal systems (3.4) for $n \geq 0$ yields the nonlinearized spatial system:

$$\begin{cases} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_x = U(f, \lambda_j)\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix},\ j = 1, 2, \cdots, N, \\ \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix}_x = -U^T(f, \lambda_j)\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix},\ j = 1, 2, \cdots, N; \end{cases} \tag{3.8}$$

and the nonlinearized temporal systems for $n \geq 0$:

$$\left\{\begin{array}{l} \left(\begin{array}{c} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{array}\right)_{t_n} = V^{(n)}(f,\lambda_j)\left(\begin{array}{c} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{array}\right),\ j=1,2,\cdots,N, \\ \left(\begin{array}{c} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{array}\right)_{t_n} = -(V^{(n)})^T(f,\lambda_j)\left(\begin{array}{c} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{array}\right),\ j=1,2,\cdots,N. \end{array}\right. \tag{3.9}$$

It is obvious that (3.8) is a system of ordinary differential equations and (3.9) is a hierarchy of partial differential equations.

Suppose that Z is an expression depending on u and its differentials. ¿From now on we use $\tilde{Z}$ to denote the expression of Z depending on $P_i, Q_i,\ 1 \leq i \leq 3$, and their differentials after substituting (3.6) into Z, and use $\mathrm{Or}(\tilde{Z})$ to denote the expression of $\tilde{Z}$ only depending on $P_i, Q_i,\ 1 \leq i \leq 3$, themselves after substituting (3.8) into $\tilde{Z}$ sufficiently many times. Therefore (3.9) may be transformed into the following systems for $n \geq 0$:

$$\left\{\begin{array}{l} \left(\begin{array}{c} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{array}\right)_{t_n} = \mathrm{Or}(V^{(n)}(f,\lambda_j))\left(\begin{array}{c} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{array}\right),\ j=1,2,\cdots,N, \\ \left(\begin{array}{c} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{array}\right)_{t_n} = -(\mathrm{Or}(V^{(n)}(f,\lambda_j))^T\left(\begin{array}{c} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{array}\right),\ j=1,2,\cdots,N, \end{array}\right. \tag{3.10}$$

which are all ordinary differential equations with an independent variable t_n because the matrices $\mathrm{Or}(V^{(n)}(f,\lambda_j)),\ n \geq 0,\ 1 \leq j \leq N$, only depend on $P_i, Q_i,\ 1 \leq i \leq 3$.

We would like to discuss the integrability on the nonlinearized spatial system (3.8) and the nonlinearized temporal systems (3.10) for $n \geq 0$ in the Liouville sense [21]. We shall utilize the symplectic structure ω^2 on $\mathbb{R}^{6N}$

$$\omega^2 = \sum_{i=0}^{3}\sum_{j=0}^{N} \mu_j d\phi_{ij} \wedge d\psi_{ij} = \sum_{i=0}^{3}(BdP_i) \wedge dQ_i,$$

by which one can define the corresponding Poisson bracket for two functions F, G defined over the phase space $\mathbb{R}^{6N}$

$$\begin{aligned} \{F,G\} &= \omega^2(IdG, IdF) = \omega^2(X_G, X_F) \\ &= \sum_{i=1}^{3}\sum_{j=1}^{N}\mu_j^{-1}\left(\frac{\partial F}{\partial \psi_{ij}}\frac{\partial G}{\partial \phi_{ij}} - \frac{\partial F}{\partial \phi_{ij}}\frac{\partial G}{\partial \psi_{ij}}\right) \\ &= \sum_{i=1}^{3}\left(< \frac{\partial F}{\partial Q_i}, B^{-1}\frac{\partial G}{\partial P_i} > - < \frac{\partial F}{\partial P_i}, B^{-1}\frac{\partial G}{\partial Q_i} >\right), \end{aligned} \tag{3.11}$$

where $IdH = X_H$ denotes the Hamiltonian vector field with energy H determined by

$$\omega^2(X, IdH) = \omega^2(X, X_H) = dH(X), \ X \in T(\mathbb{R}^{6N}),$$

and the corresponding Hamiltonian system with the Hamiltonian function H

$$\dot{x} = IdH(x) = dx(IdH) = \omega^2\{IdH, Idx\} = \{x, H\}, \ x \in \mathbb{R}^{6N}, \tag{3.12}$$

which possesses an explicit formulation

$$\dot{P}_i = -B^{-1}\frac{\partial H}{\partial Q_i}, \ \dot{Q}_i = B^{-1}\frac{\partial H}{\partial P_i}, \ i = 1, 2, 3. \tag{3.13}$$

Note that there are some authors who use the other Poisson bracket $\{F, G\} = \omega^2(X_F, X_G)$. As remarked by Carroll [9], it doesn't matter of course but each type has many proponents and hence one must be careful of minus signs in reading various sources. The notation we accept here is the Arnold's one [4].

Theorem 3.1 *The following functions*

$$\bar{F}_j = \sum_{i=1}^{3} \phi_{ij}\psi_{ij}, \ 1 \le j \le N, \tag{3.14}$$

are all integrals of motion for the nonlinearized spatial system (3.8). Moreover they are in involution under the Poisson bracket (3.11) and independent over the region

$$\Omega = \{\mathbb{R}^{6N} \,|\, \phi_{ij}, \psi_{ij} \in \mathbb{R}, \ \sum_{i=1}^{3}(\phi_{ij}^2 + \psi_{ij}^2) \neq 0, \ 1 \le j \le N\}.$$

Proof: Let

$$\bar{V}(\lambda_j) = (\phi_{kj}\psi_{lj})_{k,l=1,2,3}\,, \ 1 \le j \le N.$$

We can first find that

$$\bar{F}_j = \mathrm{tr}(\bar{V}(\lambda_j)).$$

On the other hand, by Lemma 3.1 we know that $\bar{V}(\lambda_j)$ satisfies

$$\bar{V}(\lambda_j)_x = [U(\tilde{u}, \lambda_j), \bar{V}(\lambda_j)]$$

when (3.8) holds, and thus

$$\begin{aligned} \bar{F}_{jx} &= (\mathrm{tr}(\bar{V}(\lambda_j)))_x = \mathrm{tr}(\bar{V}(\lambda_j))_x \\ &= \mathrm{tr}[U(\tilde{u}, \lambda_j), \bar{V}(\lambda_j)] = 0, \end{aligned}$$

which shows that $\bar{F}_j$, $1 \le j \le N$, are all integrals of motion for the nonlinearized spatial system (3.8). In addition, it is very easy to prove that

$$\{\bar{F}_k, \bar{F}_l\} = 0, \ 1 \le k, l \le N,$$

which means $\bar{F}_j,\ 1\le j\le N$, are in involution. It is also obvious that grad$\bar{F}_j,\ 1\le j\le N$, are everywhere linear independent over Ω by observing that

$$\left(\frac{\partial \bar{F}_k}{\partial \phi_{il}}\right)_{k,l=1,\cdots,N} = \begin{pmatrix} \psi_{i1} & & & 0 \\ & \psi_{i2} & & \\ & & \ddots & \\ 0 & & & \psi_{iN} \end{pmatrix},\ i=1,2,3,$$

$$\left(\frac{\partial \bar{F}_k}{\partial \psi_{il}}\right)_{k,l=1,\cdots,N} = \begin{pmatrix} \phi_{i1} & & & 0 \\ & \phi_{i2} & & \\ & & \ddots & \\ 0 & & & \phi_{iN} \end{pmatrix},\ i=1,2,3.$$

The proof is completed. ▮

Throughout our paper, we assume that

$$A=\mathrm{diag}(\lambda_1,\lambda_2,\cdots,\lambda_N). \tag{3.15}$$

If all elements $Z_{ij},\ 1\le i,j\le s$ of a given matrix $Z=(Z_{ij})_{s\times s}$ are polynomials in λ, i. e. $Z_{ij}=\sum_{k=0}^m Z_{ijk}\lambda^k$, for convenience of presentation, we define a new matrix called $M_A(Z)$ as follows

$$M_A(Z)=(\sum_{k=0}^m Z_{ijk}A^k)_{sN\times sN}. \tag{3.16}$$

Moreover we often accept compact forms, for example,

$$\frac{\partial}{\partial P_i}=(\frac{\partial}{\partial \phi_{i1}},\cdots,\frac{\partial}{\partial \phi_{iN}})^T,\ \{P_i,H\}=(\{\phi_{i1},H\},\cdots,\{\phi_{iN},H\})^T,\ i=1,2,3.$$

Theorem 3.2 *We have the explicit integrals of motion for the nonlinearized spatial system (3.8):*

$$\left\{\begin{array}{l} F_1=-8(<P_1,BQ_1>-<P_3,BQ_3>), \\ F_m=4\sum_{i=1}^{m-1}[(<A^{i-1}P_1,BQ_1>-<A^{i-1}P_3,BQ_3>)\times \\ \qquad (<A^{m-i-1}P_1,BQ_1>-<A^{m-i-1}P_3,BQ_3>) \\ \qquad +2(<A^{i-1}P_1,BQ_2>+<A^{i-1}P_2,BQ_3>)\times \\ \qquad (<A^{m-i-1}P_2,BQ_1>+<A^{m-i-1}P_3,BQ_2>)] \\ \qquad -8(<A^{m-1}P_1,BQ_1>-<A^{m-1}P_3,BQ_3>),\ m\ge 2, \end{array}\right. \tag{3.17}$$

where P_i, Q_i, B are defined by (3.7). Moreover they constitute an involutive system together with $\bar{F}_j,\ 1\le j\le N$, under the Poisson bracket (3.11), i.e.

$$\{F_k,F_l\}=\{F_m,\bar{F}_j\}=0,\ m,k,l\ge 1,\ 1\le j\le N.$$

Proof: We assume that

$$
\begin{aligned}
&\hat{q}=\sqrt{2}(<P_1,BQ_2>+<P_2,BQ_3>),\ \hat{r}=\sqrt{2}(<P_2,BQ_1>+<P_3,BQ_2>);\\
&\hat{a}_0=-1,\ \hat{b}_0=\hat{c}_0=0;\\
&\begin{cases}\hat{a}_{i+1}=<A^iP_1,BQ_1>-<A^iP_3,BQ_3>,\ i\geq 0,\\ \hat{b}_{i+1}=\sqrt{2}(<A^iP_1,BQ_2>+<A^iP_2,BQ_3>),\ i\geq 0,\\ \hat{c}_{i+1}=\sqrt{2}(<A^iP_2,BQ_1>+<A^iP_3,BQ_2>),\ i\geq 0.\end{cases}
\end{aligned}
\tag{3.18}
$$

Further we choose that

$$\hat{U}=\begin{pmatrix}-2\lambda & \sqrt{2}\hat{q} & 0\\ \sqrt{2}\hat{r} & 0 & \sqrt{2}\hat{q}\\ 0 & \sqrt{2}\hat{r} & 2\lambda\end{pmatrix},\ \hat{V}=\begin{pmatrix}2\hat{a} & \sqrt{2}\hat{b} & 0\\ \sqrt{2}\hat{c} & 0 & \sqrt{2}\hat{b}\\ 0 & \sqrt{2}\hat{c} & -2\hat{a}\end{pmatrix},$$

where $\hat{a},\ \hat{b}$ and $\hat{c}$ are defined by

$$\hat{a}=\sum_{i=0}^{\infty}\hat{a}_i\lambda^{-i},\ \hat{b}=\sum_{i=0}^{\infty}\hat{b}_i\lambda^{-i},\ \hat{c}=\sum_{i=0}^{\infty}\hat{c}_i\lambda^{-i}.$$

It may be shown that when the nonlinearized spatial system (3.8) holds, we have

$$\hat{V}_x=[\hat{U},\hat{V}],\ \text{i.e. } \hat{a}_x=\hat{q}\hat{c}-\hat{r}\hat{b},\ \hat{b}_x=-2\lambda\hat{b}-2\hat{q}\hat{a},\ \hat{c}_x=2\lambda\hat{c}+2\hat{r}\hat{a}.$$

Therefore we can compute that

$$\hat{F}_x:=(\frac{1}{2}\mathrm{tr}(\hat{V}^2))_x=\frac{1}{2}\mathrm{tr}(\hat{V}^2)_x=\frac{1}{2}\mathrm{tr}[\hat{U},\hat{V}^2]=0.$$

On the other hand, we have

$$\hat{F}=4(\hat{a}^2+\hat{b}\hat{c})=\sum_{m=0}^{\infty}F_m\lambda^{-m},\ F_0=4,\ F_m=4\sum_{i=0}^{m}(\hat{a}_i\hat{a}_{m-i}+\hat{b}_i\hat{c}_{m-i}),\ m\geq 1.$$

Hence $F_m,\ m\geq 1$, are all integrals of motion for the nonlinearized spatial system (3.8).

Now we turn to the involutivity of integrals of motion. We take

$$\hat{V}^{(n)}(\lambda)=(\lambda^n\hat{V})_+$$

and construct a temporal system for $n\geq 0$

$$\begin{pmatrix}P_1\\P_2\\P_3\end{pmatrix}_{t_n}=M_A(\hat{V}^{(n)})\begin{pmatrix}P_1\\P_2\\P_3\end{pmatrix},\ \begin{pmatrix}Q_1\\Q_2\\Q_3\end{pmatrix}_{t_n}=-(M_A(\hat{V}^{(n)}))^T\begin{pmatrix}Q_1\\Q_2\\Q_3\end{pmatrix},\tag{3.19}$$

where $M_A(\hat{V}^{(n)})$, $n\geq 0$, are determined in the way of (3.16). We can first prove that when this system (3.19) holds, we have

$$(\hat{V}(\lambda))_{t_n}=[\hat{V}^{(n)}(\lambda),\hat{V}(\lambda)].$$

Therefore $F_m,\ m \geq 1$, are also integrals of motion for the system (3.19). Secondly, we can verify that

$$\begin{pmatrix} B^{-1}\frac{\partial \hat F}{\partial P_1} \\ B^{-1}\frac{\partial \hat F}{\partial P_2} \\ B^{-1}\frac{\partial \hat F}{\partial P_3} \end{pmatrix} = \begin{pmatrix} B^{-1}\mathrm{tr}(\hat V \frac{\partial}{\partial P_1}\hat V) \\ B^{-1}\mathrm{tr}(\hat V \frac{\partial}{\partial P_2}\hat V) \\ B^{-1}\mathrm{tr}(\hat V \frac{\partial}{\partial P_3}\hat V) \end{pmatrix} = \sum_{m=0}^{\infty}(M_A(\hat V^{(m)}))^T \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} \lambda^{-m-1},$$

$$\begin{pmatrix} B^{-1}\frac{\partial \hat F}{\partial Q_1} \\ B^{-1}\frac{\partial \hat F}{\partial Q_2} \\ B^{-1}\frac{\partial \hat F}{\partial Q_3} \end{pmatrix} = \begin{pmatrix} B^{-1}\mathrm{tr}(\hat V \frac{\partial}{\partial Q_1}\hat V) \\ B^{-1}\mathrm{tr}(\hat V \frac{\partial}{\partial Q_2}\hat V) \\ B^{-1}\mathrm{tr}(\hat V \frac{\partial}{\partial Q_3}\hat V) \end{pmatrix} = \sum_{m=0}^{\infty}(M_A(\hat V^{(m)})) \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix} \lambda^{-m-1}.$$

These two equalities show that the system (3.19) for $n \geq 0$ are all Hamiltonian systems with Hamiltonian functions $-F_{n+1}$. Therefore

$$\{F_{m+1}, -F_{n+1}\} = \frac{d}{dt_n}F_{m+1} = 0,\ m, n \geq 0,$$

which shows the involutivity of $F_m,\ m \geq 1$. In addition, it is easy to get that

$$\{F_k, \bar F_l\} = \sum_{i=0}^{3}\mu_l^{-1}(\frac{\partial F_k}{\partial \psi_{il}}\psi_{il} - \frac{\partial F_k}{\partial \phi_{il}}\phi_{il}) = 0,\ k \geq 1,\ 1 \leq l \leq N,$$

noting the particular form of $F_k,\ k \geq 1$. The proof is finished. ▮

Because we have $\bar V^2(\lambda_j) = \mathrm{tr}(\bar V(\lambda_j))\bar V(\lambda_j),\ 1 \leq j \leq N,\ \hat V^3 = \frac{1}{2}\mathrm{tr}(\hat V^2)\hat V$, we cannot obtain new integrals of motion of the nonlinearized spatial system (3.8) from the trace of other power of $\bar V(\lambda_j)$ and $\hat V$. In addition to this, it is interesting to observe that the determinants of the matrices $\bar V(\lambda_j),\ 1 \leq j \leq N$, and $\hat V$ are all zero.

The nonlinearized spatial system is easily rewritten as an Hamiltonian system

$$P_{ix} = \{P_i, H\} = -B^{-1}\frac{\partial H}{\partial Q_i},\ Q_{ix} = \{Q_i, H\} = B^{-1}\frac{\partial H}{\partial P_i},\ i = 1, 2, 3$$

with the Hamiltonian function

$$\begin{aligned} &H = 2(< AP_1, BQ_1 > - < AP_3, BQ_3 >) \\ &-2(< P_1, BQ_2 > + < P_2, BQ_3 >)(< P_2, BQ_1 > + < P_3, BQ_2 >) = -\tfrac{1}{4}F_2 + \tfrac{1}{64}F_1^2. \end{aligned}$$

Thus it possesses the following $3N$ involutive integrals of motion

$$\bar F_j,\ 1 \leq j \leq N, F_m,\ 1 \leq m \leq 2N.$$

In some special cases, they may be shown to be independent at least on certain region.

Theorem 3.3 *When $N=1,2$, the determinant of the matrix*

$$D(N)=\left(\begin{array}{ccc}(grad_{P_1}\bar{F}_1)^T & (grad_{P_2}\bar{F}_1)^T & (grad_{P_3}\bar{F}_1)^T\\ \vdots & \vdots & \vdots\\ (grad_{P_1}\bar{F}_N)^T & (grad_{P_2}\bar{F}_N)^T & (grad_{P_3}\bar{F}_N)^T\\ (grad_{P_1}F_1)^T & (grad_{P_2}F_1)^T & (grad_{P_3}F_1)^T\\ \vdots & \vdots & \vdots\\ (grad_{P_1}F_{2N})^T & (grad_{P_2}F_{2N})^T & (grad_{P_3}F_{2N})^T\end{array}\right)$$

with $grad_{P_i}G=(\frac{\partial G}{\partial\phi_{i1}},\cdots,\frac{\partial G}{\partial\phi_{iN}})^T,\ 1\le i\le 3$, is not always zero. Thus the integrals of motion $\bar{F}_j,\ 1\le j\le N, F_m,\ 1\le m\le 2N$, are independent at least on some region.

Proof: We have

$$\begin{aligned}D(1)\ &=\ (16\phi_{11}\psi_{11}\psi_{21}^3+16\phi_{31}\psi_{21}^3\psi_{31}-32\phi_{11}\psi_{11}^2\psi_{21}\psi_{31}\\ &\quad +32\phi_{21}\psi_{11}\psi_{21}^2\psi_{31}-32\phi_{31}\psi_{11}\psi_{21}\psi_{31}^2-64\phi_{21}\psi_{11}^2\psi_{31}^2)\mu_1^3\end{aligned}$$

and

$$D(2)=(-256\lambda_1^3+256\lambda_2^3-768\lambda_1\lambda_2^2+768\lambda_1^2\lambda_2)\mu_1^3\mu_2^4,$$

when we choose

$$\begin{aligned}&\phi_{11}=1,\ \phi_{12}=0,\ \phi_{21}=0,\ \phi_{22}=-1,\ \phi_{31}=0,\ \phi_{32}=0,\\ &\psi_{11}=0,\ \psi_{12}=-1,\ \psi_{21}=1,\ \psi_{22}=0,\ \psi_{31}=0,\ \psi_{32}=-1.\end{aligned}$$

These are consequences of computation by the computer algebra system MuPAD. But they may also be shown by some direct computation. The proof is completed. ▮

According to the above theorem, the nonlinearized spatial system (3.8) is Liouville integrable on some region of the phase space, when $N=1,2$.

4 Involutive solutions

The aim of this section is to discuss some properties on the nonlinearized spatial system (3.8) and the nonlinearized spatial system (3.10) and to establish a kind of involutive solutions with separated variables for AKNS soliton equations.

Lemma 4.1 *When $\phi=(\phi_1,\phi_2,\phi_3)^T$ and $\psi=(\psi_1,\psi_2,\psi_3)^T$ satisfy the spectral problem (2.1) and the adjoint spectral problem (2.2), we have*

$$L\left(\begin{array}{c}\phi_2\psi_1+\phi_3\psi_2\\ \phi_1\psi_2+\phi_2\psi_3\end{array}\right)=\lambda\left(\begin{array}{c}\phi_2\psi_1+\phi_3\psi_2\\ \phi_1\psi_2+\phi_2\psi_3\end{array}\right)+I\left(\begin{array}{c}r\\ q\end{array}\right),\tag{4.1}$$

where I is an integral of motion for (2.1) and (2.2).

Proof: ¿From the spectral problem (2.1) and the adjoint spectral problem (2.2), we can find that

$$\partial(\phi_1\psi_1 - \phi_3\psi_3) = \sqrt{2}q(\phi_2\psi_1 + \phi_3\psi_2) - \sqrt{2}r(\phi_1\psi_2 + \phi_2\psi_3).$$

This yields

$$\partial^{-1}[-q(\phi_2\psi_1 + \phi_3\psi_2) + r(\phi_1\psi_2 + \phi_2\psi_3)] = -\frac{1}{\sqrt{2}}(\phi_1\psi_1 - \phi_3\psi_3) + I,$$

where I is an integrals of motion for (2.1) and (2.2). The relation (4.1) follows from the above equality. ▮

We recall that $\widetilde{Z}$ denotes the expression of Z depending on P_i, Q_i, $1 \le i \le 3$, and their differentials after the substitution of (3.6) and into Z, and that $\mathrm{Or}(\widetilde{Z})$ denotes the expression of $\widetilde{Z}$ only depending on P_i, Q_i, $1 \le i \le 3$, themselves after the substitution of (3.8) into $\widetilde{Z}$ sufficiently many times. A general result on $\tilde{a}_i, \tilde{b}_i, \tilde{c}_i$, $i \ge 1$, is given in the following theorem.

Theorem 4.1 *We have the explicit expressions for $\tilde{a}_m$, $\tilde{b}_m$, $\tilde{c}_m, m \ge 1$:*

$$\begin{aligned} \tilde{a}_{m+1} &= \sum_{i=0}^{m} I_i(< A^{m-i}P_1, BQ_1 > - < A^{m-i}P_3, BQ_3 >) - I_{m+1},\ m \ge 0, && (4.2) \\ \tilde{b}_{m+1} &= \sqrt{2}\sum_{i=0}^{m} I_i(< A^{m-i}P_1, BQ_2 > + < A^{m-i}P_2, BQ_3 >),\ m \ge 0, && (4.3) \\ \tilde{c}_{m+1} &= \sqrt{2}\sum_{i=0}^{m} I_i(< A^{m-i}P_2, BQ_1 > + < A^{m-i}P_3, BQ_2 >),\ m \ge 0, && (4.4) \end{aligned}$$

provided that the nonlinearized spatial system (3.8) is satisfied. Here I_m, $m \ge 0$, are defined by

$$I_0 = 1,\ I_m = \sum_{n=1}^{m} d_n \sum_{\substack{i_1+\cdots+i_n=m \\ i_1,\cdots,i_n \ge 1}} F_{i_1}\cdots F_{i_n},\ m \ge 1, \tag{4.5}$$

where the constants d_n, $n \ge 0$, are determined recursively by

$$\begin{cases} d_1 = -\frac{1}{8},\ d_2 = \frac{3}{128}, \\ d_n = -\frac{1}{2}\sum_{i=1}^{n-1} d_i d_{n-i} - \frac{1}{4}d_{n-1} - \frac{1}{8}\sum_{i=1}^{n-2} d_i d_{n-i-1},\ n \ge 3, \end{cases} \tag{4.6}$$

and the functions F_m, $m \ge 1$, are given by (3.17).

Proof: By the recursion relation (2.7) and Lemma (4.1), we can obtain that

$$\begin{aligned} \begin{pmatrix} \tilde{c}_{m+1} \\ \tilde{b}_{m+1} \end{pmatrix} &= \tilde{L}^m \begin{pmatrix} \tilde{r} \\ \tilde{q} \end{pmatrix} = \tilde{L}^m \begin{pmatrix} \sqrt{2}(< P_2, BQ_1 > + < P_3, BQ_2 >) \\ \sqrt{2}(< P_1, BQ_2 > + < P_2, BQ_3 >) \end{pmatrix} \\ &= \sum_{i=0}^{m} I_i \begin{pmatrix} \sqrt{2}(< A^{m-i}P_2, BQ_1 > + < A^{m-i}P_3, BQ_2 >) \\ \sqrt{2}(< A^{m-i}P_1, BQ_2 > + < A^{m-i}P_2, BQ_3 >) \end{pmatrix},\ m \ge 0, \end{aligned}$$

where $I_0=1$ and $I_i,\ 1\le i\le m$, are integrals of motion for the nonlinearized spatial system (3.8). Now we compute $\tilde a_{m+1},\ m\ge 0$, by $\tilde b_{mx}=-2\tilde b_{m+1}-2\tilde q\,\tilde a_m,\ m\ge 0$. Noting (3.8), we have for $m\ge 0$

$$\begin{aligned}\tilde b_{mx} &= \sum_{i=0}^{m-1}\sqrt{2}I_i(< A^{m-i-1}P_{1x},BQ_2> + < A^{m-i-1}P_1,BQ_{2x}> \\ &\qquad + < A^{m-i-1}P_{2x},BQ_3> + < A^{m-i-1}P_2,BQ_{3x}>) \\ &= \sum_{i=0}^{m-1}\sqrt{2}I_i(< -2A^{m-i}P_1,BQ_2> + < A^{m-i-1}P_1,-\sqrt{2}\tilde q BQ_1> \\ &\qquad + < \sqrt{2}\tilde q A^{m-i-1}P_3,BQ_3> + < A^{m-i-1}P_2,-2ABQ_3>) \\ &= -2\tilde b_{m+1}-2\tilde q\Big(\sum_{i=0}^{m-1}I_i(< A^{m-i-1}P_1,BQ_1> - < A^{m-i-1}P_3,BQ_3>)-I_m\Big),\end{aligned}$$

from which (4.2) follows.

In the following we determine the integrals of motion $I_m,\ m\ge 0$, by a relation

$$\tilde a^2+\tilde b\tilde c=1,\ \tilde a=\sum_{i=0}^{\infty}\tilde a_i\lambda^{-i},\ \tilde b=\sum_{i=0}^{\infty}\tilde b_i\lambda^{-i},\ \tilde c=\sum_{i=0}^{\infty}\tilde c_i\lambda^{-i},$$

which gives rise to

$$2\tilde a_m=\sum_{i=1}^{m-1}(\tilde a_i\tilde a_{m-i}+\tilde b_i\tilde c_{m-i}),\ m\ge 2. \tag{4.7}$$

First from $\tilde a_1=0$ we have

$$I_1=< P_1,BQ_1> - < P_3,BQ_3> = -\frac{1}{8}F_1,$$

which shows $d_1=-\frac18$. Now we suppose $m\ge 2$. At this moment, we have by (4.7)

$$\begin{aligned}&2\sum_{i=0}^{m-1}I_i\hat a_{m-i}-2I_m \\ =\ &\sum_{i=1}^{m-1}\Big(\sum_{k=0}^{i-1}I_k\hat a_{i-k}-I_i\Big)\Big(\sum_{l=0}^{m-i-1}I_l\hat a_{m-i-l}-I_{m-i}\Big) \\ &+\sum_{i=1}^{m-1}\sum_{k=0}^{i-1}I_k\hat b_{i-k}\sum_{l=0}^{m-i-1}I_l\hat c_{m-i-l},\ m\ge 2,\end{aligned}$$

where $\hat a_i,\ \hat b_i,\ \hat c_i,\ i\ge 1$, are given by (3.18). After interchanging the summing in the above equality, i.e.

$$\sum_{i=1}^{m-1}\sum_{k=0}^{i-1}=\sum_{k=0}^{m-2}\sum_{i=k+1}^{m-1},\ \sum_{i=1}^{m-1}\sum_{l=0}^{m-i-1}=\sum_{l=0}^{m-2}\sum_{i=1}^{m-1-l},\ \sum_{i=1}^{m-1}\sum_{k=0}^{i-1}\sum_{l=0}^{m-i-1}=\sum_{k=0}^{m-2}\sum_{l=0}^{(m-2)-k}\sum_{i=k+1}^{m-(l+1)},$$

we may arrive at

$$
\begin{aligned}
-2I_m &= \sum_{i=1}^{m-1} I_i I_{m-i} + \sum_{k=0}^{m-2} \sum_{l=0}^{(m-2)-k} \sum_{i=k+1}^{m-(l+1)} I_k I_l (\hat{a}_{i-k}\hat{a}_{m-i-l} + \hat{b}_{i-k}\hat{c}_{m-i-l}) \\
&\quad - \sum_{k=0}^{m-2} \sum_{i=k+1}^{m-1} I_k I_{m-i}\hat{a}_{i-k} - \sum_{l=0}^{m-2} \sum_{i=1}^{m-l-1} I_l I_i \hat{a}_{m-i-l} - 2\sum_{i=0}^{m-1} I_i \hat{a}_{m-i} \\
&:= B_1 + B_2 + B_3 + B_4 + B_5,\ m \geq 2. \qquad (4.8)
\end{aligned}
$$

Further we have

$$
\begin{aligned}
B_3 &= -\sum_{k=0}^{m-2} \Big(\sum_{i=k+2}^{m} + \sum_{i=k+1} - \sum_{i=m} \Big) I_k I_{m-i} \hat{a}_{i-k} \\
&= -\sum_{k=0}^{m-2} \sum_{i=k+2}^{m} I_k I_{m-i}\hat{a}_{i-k} - \sum_{k=0}^{m-2} I_k I_{m-k-1}\hat{a}_1 + \sum_{k=0}^{m-2} I_k \hat{a}_{m-k} \\
&= -\sum_{k=0}^{m-2} \sum_{l=0}^{(m-2)-k} I_k I_l \hat{a}_{m-(k+l)} - \sum_{k=0}^{m-2} I_k I_{m-k-1}\hat{a}_1 + \sum_{k=0}^{m-2} I_k \hat{a}_{m-k},
\end{aligned}
$$

$$
\begin{aligned}
B_4 &= -\sum_{l=0}^{m-2} \Big(\sum_{i=0}^{m-l-2} - \sum_{i=0} + \sum_{i=m-l-1} \Big) I_l I_i \hat{a}_{m-i-l} \\
&= -\sum_{k=0}^{m-2} \sum_{l=0}^{(m-2)-k} I_k I_l \hat{a}_{m-(k+l)} + \sum_{l=0}^{m-2} I_l \hat{a}_{m-l} - \sum_{l=0}^{m-2} I_l I_{m-l-1}\hat{a}_1.
\end{aligned}
$$

Therefore the latter three terms in the right hand side of (4.8) becomes

$$
B_3 + B_4 + B_5 = -2\sum_{k=0}^{m-2} \sum_{l=0}^{(m-2)-k} I_k I_l \hat{a}_{m-(k+l)} + \frac{1}{4} \sum_{\substack{k+l=m-1 \\ k,l\geq 0}} I_k I_l F_1.
$$

In this way, from (4.8) we obtain

$$
\begin{aligned}
I_m &= -\frac{1}{2}\sum_{i=1}^{m-1} I_i I_{m-i} - \frac{1}{8}\sum_{k=0}^{m-2} \sum_{l=0}^{(m-2)-k} I_k I_l F_{m-(k+l)} - \frac{1}{8} \sum_{\substack{k+l=m-1 \\ k,l\geq 0}} I_k I_l F_1 \\
&= -\frac{1}{2}\sum_{i=1}^{m-1} I_i I_{m-i} - \frac{1}{8} \sum_{\substack{k+l\leq m-1 \\ k,l\geq 0}} I_k I_l F_{m-(k+l)},\ m \geq 2, \qquad (4.9)
\end{aligned}
$$

by which we can determine any I_m, $m \geq 2$, starting with $I_1 = -\frac{1}{8}F_1$. It is not difficult to find a homogeneous property among the terms of (4.9). Thus we may assume that

$$
I_m = \sum_{n=1}^{m} d_n \sum_{\substack{i_1+\cdots+i_n=m \\ i_1,\cdots,i_n\geq 1}} F_{i_1}\cdots F_{i_n},\ m \geq 2.
$$

In general, the coefficients d_n, $1\le n\le m$, should depend on m. But the following deduction implies that this assumption is possible. First from (4.9) we easily have $I_2=\frac{3}{128}F_1^2-\frac{1}{8}F_2$, which leads to $d_2=\frac{3}{128}$. When $m\ge 3$, (4.9) becomes

$$I_m=-\frac{1}{2}\sum_{i=1}^{m-1}I_iI_{m-i}-\frac{1}{8}F_m-\frac{1}{4}\sum_{k=1}^{m-1}I_kF_{m-k}-\frac{1}{8}\sum_{\substack{k+l\le m-1\\ k,l\ge 1}}I_kI_lF_{m-(k+l)},\qquad (4.10)$$

in which the coefficients of the F_1^m yields the recursion relation (4.6). In what follows, we want to prove that I_m, $m\ge 0$, determined above satisfy the relation (4.10), indeed. This may be shown by combining the following three equalities. First we have

$$\begin{aligned}\sum_{i=1}^{m-1}I_iI_{m-i}&=\sum_{i=1}^{m-1}\sum_{k=1}^{i}d_k\sum_{\substack{i_1+\cdots+i_k=i\\ i_1,\cdots,i_k\ge 1}}F_{i_1}\cdots F_{i_k}\sum_{l=1}^{m-i}d_l\sum_{\substack{j_1+\cdots+j_l=m-i\\ j_1,\cdots,j_l\ge 1}}F_{j_1}\cdots F_{j_l}\\
&=\sum_{k=1}^{m-1}\sum_{i=k}^{m-1}\sum_{l=1}^{m-i}d_kd_l\sum_{\substack{i_1+\cdots+i_k=i\\ i_1,\cdots,i_k\ge 1}}F_{i_1}\cdots F_{i_k}\sum_{\substack{j_1+\cdots+j_l=m-i\\ j_1,\cdots,j_l\ge 1}}F_{j_1}\cdots F_{j_l}\\
&=\sum_{k=1}^{m-1}\sum_{l=1}^{m-k}\sum_{i=k}^{m-l}d_kd_l\sum_{\substack{i_1+\cdots+i_k=i\\ i_1,\cdots,i_k\ge 1}}F_{i_1}\cdots F_{i_k}\sum_{\substack{j_1+\cdots+j_l=m-i\\ j_1,\cdots,j_l\ge 1}}F_{j_1}\cdots F_{j_l}\\
&=\sum_{n=2}^{m}\sum_{\substack{k+l=n\\ k,l\ge 1}}d_kd_l\sum_{\substack{i_1+\cdots+i_n=m\\ i_1,\cdots,i_n\ge 1}}F_{i_1}\cdots F_{i_n}.\end{aligned}$$

Similarly we can get the other two equalities

$$\sum_{k=1}^{m-1}I_kF_{m-k}=\sum_{n=2}^{m}d_{n-1}\sum_{\substack{i_1+\cdots+i_n=m\\ i_1,\cdots,i_n\ge 1}}F_{i_1}\cdots F_{i_n},$$

$$\sum_{\substack{k+l\le m-1\\ k,l\ge 1}}I_kI_lF_{m-(k+l)}=\sum_{n=2}^{m-1}\sum_{\substack{i+j=n\\ i,j\ge 1}}d_id_j\sum_{\substack{i_1+\cdots+i_{n+1}=m\\ i_1,\cdots,i_{n+1}\ge 1}}F_{i_1}\cdots F_{i_{n+1}}.$$

Therefore the proof is finished. ▮

Theorem 4.2 *If P_i and Q_i, $1\le i\le 3$, solve the nonlinearized spatial system (3.8), then there exist N integrals of motion α_i, $0\le i\le N-1$, of (3.8) such that*

$$q=\sqrt{2}(<P_1,BQ_2>+<P_2,BQ_3>),\ \ r=\sqrt{2}(<P_2,BQ_1>+<P_3,BQ_2>)$$

solve the following N-th order stationary AKNS equation

$$K_N + \sum_{i=0}^{N-1} \alpha_i K_i = 0,$$

where K_i, $0 \le i \le N$, are defined by (2.10).

Proof: Noting that the expressions (4.3) and (4.4) of $\tilde{b}_{i+1}$, $\tilde{c}_{i+1}$, $i \ge 0$, we can compute that

$$\begin{aligned}
\sum_{i=0}^{N} \alpha_i \widetilde{K}_i &= \sum_{i=0}^{N} \alpha_i J \begin{pmatrix} \tilde{c}_{i+1} \\ \tilde{b}_{i+1} \end{pmatrix} \\
&= J \sum_{i=0}^{N} \alpha_i \sum_{j=0}^{i} I_j \begin{pmatrix} \sqrt{2}(< A^{i-j}P_2, BQ_1 > + < A^{i-j}P_3, BQ_2 >) \\ \sqrt{2}(< A^{i-j}P_1, BQ_2 > + < A^{i-j}P_2, BQ_3 >) \end{pmatrix} \\
&= \sqrt{2} J \sum_{i=0}^{N} \alpha_i \sum_{j=0}^{i} I_j \sum_{k=1}^{N} \lambda_k^{i-j} \mu_k \begin{pmatrix} \phi_{2k}\psi_{1k} + \phi_{3k}\psi_{2k} \\ \phi_{1k}\psi_{2k} + \phi_{2k}\psi_{3k} \end{pmatrix} \\
&= \sqrt{2} J \sum_{k=1}^{N} \mu_k \Big(\sum_{i=0}^{N} \sum_{j=0}^{i} \alpha_i I_j \lambda_k^{i-j} \Big) \begin{pmatrix} \phi_{2k}\psi_{1k} + \phi_{3k}\psi_{2k} \\ \phi_{1k}\psi_{2k} + \phi_{2k}\psi_{3k} \end{pmatrix} \\
&= \sqrt{2} J \sum_{k=1}^{N} \mu_k \sum_{l=0}^{N} \Big(\sum_{i=l}^{N} \alpha_i I_{i-l} \Big) \lambda_k^{l} \begin{pmatrix} \phi_{2k}\psi_{1k} + \phi_{3k}\psi_{2k} \\ \phi_{1k}\psi_{2k} + \phi_{2k}\psi_{3k} \end{pmatrix}.
\end{aligned}$$

Secondly we set

$$G(\lambda) = \prod_{i=1}^{N} (\lambda - \lambda_i) = \sum_{i=0}^{N} \beta_i \lambda^i = \lambda^N + \sum_{i=0}^{N-1} \beta_i \lambda^i.$$

Let us now choose

$$\sum_{i=l}^{N} \alpha_i I_{i-l} = \beta_l, \ 0 \le l \le N,$$

which determines recursively

$$\alpha_N = \beta_N = 1, \ \alpha_l = \beta_l - \sum_{i=l+1}^{N} \alpha_i I_{i-l}, \ 0 \le l \le N-1,$$

due to $I_0 = 1$. The α_i, $0 \le i \le N-1$, are all integrals of motion of (3.8) since they are functions of I_i, $0 \le i \le N$. Further by $G(\lambda_k) = 0$, $1 \le k \le N$, we see that $\sum_{k=0}^{N} \alpha_i \widetilde{K}_i = 0$, which completes the proof. ▮

The above theorem also implies that the potential determined by the Bargmann symmetry constraint (3.5) is a finite gap potential of the spectral problem (2.1).

Theorem 4.3 *Under the control of the nonlinearized spatial system (3.8) , the nonlinearized temporal systems (3.9) for $n \ge 0$ can also be rewritten as the Hamiltonian systems*

$$P_{it_n} = \{P_i, H_n\} = -B^{-1} \frac{\partial H_n}{\partial Q_i}, \ Q_{it_n} = \{Q_i, H_n\} = B^{-1} \frac{\partial H_n}{\partial P_i}, \quad i = 1, 2, 3 \qquad (4.11)$$

with the Hamiltonian functions

$$H_n = -\frac{1}{4}\sum_{m=0}^{n}\frac{d_m}{m+1}\sum_{\substack{i_1+\cdots+i_{m+1}=n+1\\ i_1,\cdots,i_{m+1}\geq 1}} F_{i_1}\cdots F_{i_{m+1}}\,,$$

where $d_0 = 1$ *and* F_m, $m\geq 1$, *are defined by (3.17).*

Proof: We only prove the former equality of (4.11). We know that under the control of the nonlinearized spatial system (3.8), the results in Theorem 4.1 holds. Hence we have

$$\begin{aligned} P_{1t_n} &= 2\sum_{i=0}^{n}\tilde{a}_iA^{n-i}P_1+\sqrt{2}\sum_{i=0}^{n}\tilde{b}_iA^{n-i}P_2 \\ &= -2A^nP_1+2\sum_{i=1}^{n}(\sum_{k=0}^{i-1}I_k\hat{a}_{i-k}-I_i)A^{n-i}P_1+\sqrt{2}\sum_{i=1}^{n}\sum_{k=0}^{i-1}I_k\hat{b}_{i-k}A^{n-i}P_2 \\ &= -2\sum_{k=0}^{n}I_kA^{n-k}P_1+\sum_{k=0}^{n-1}I_k\sum_{i=k+1}^{n}2\hat{a}_{i-k}A^{n-i}P_1+\sum_{k=0}^{n-1}I_k\sum_{i=k+1}^{n}\sqrt{2}\hat{b}_{i-k}A^{n-i}P_2 \\ &= \frac{1}{4}I_nB^{-1}\frac{\partial F_1}{\partial Q_1}+\frac{1}{4}\sum_{k=0}^{n-1}I_kB^{-1}\frac{\partial F_{n-k+1}}{\partial Q_1}=\frac{1}{4}\sum_{k=0}^{n}I_kB^{-1}\frac{\partial F_{n-k+1}}{\partial Q_1}, \end{aligned}$$

where $\hat{a}_i$, $\hat{b}_i$, $i\geq 1$, are given by (3.18). We further note the expression of $I_m,\ m\geq 0$, defined by (4.5) and then we may make the following performance

$$\begin{aligned} P_{1t_n} &= \frac{1}{4}I_0B^{-1}\frac{\partial F_{n+1}}{\partial Q_1}+\frac{1}{4}\sum_{k=1}^{n}\sum_{m=1}^{k}d_m\sum_{\substack{i_1+\cdots+i_m=k\\ i_1,\cdots,i_m\geq 1}}F_{i_1}\cdots F_{i_m}B^{-1}\frac{\partial F_{n-k+1}}{\partial Q_1} \\ &= \frac{1}{4}I_0B^{-1}\frac{\partial F_{n+1}}{\partial Q_1}+\frac{1}{4}\sum_{m=1}^{n}d_m\sum_{k=m}^{n}\sum_{\substack{i_1+\cdots+i_m=k\\ i_1,\cdots,i_m\geq 1}}F_{i_1}\cdots F_{i_m}B^{-1}\frac{\partial F_{n-k+1}}{\partial Q_1} \\ &= \frac{1}{4}I_0B^{-1}\frac{\partial F_{n+1}}{\partial Q_1}+\frac{1}{4}\sum_{m=1}^{n}\frac{d_m}{m+1}B^{-1}\frac{\partial}{\partial Q_1}\sum_{\substack{i_1+\cdots+i_{m+1}=n+1\\ i_1,\cdots,i_{m+1}\geq 1}}F_{i_1}\cdots F_{i_{m+1}} \\ &= \frac{1}{4}B^{-1}\frac{\partial}{\partial Q_1}\sum_{m=0}^{n}\frac{d_m}{m+1}\sum_{\substack{i_1+\cdots+i_{m+1}=n+1\\ i_1,\cdots,i_{m+1}\geq 1}}F_{i_1}\cdots F_{i_{m+1}}=-B^{-1}\frac{\partial H_n}{\partial Q_1}, \end{aligned}$$

where we have accepted $d_0=1$. The above manipulation is fulfilled for the case of $n\geq 1$. The case of $n=1$ needs only a simple calculation. Thus the former equality of (4.11) is true for $n\geq 0$. The latter equality of (4.11) may be proved similarly. The proof is completed. ∎

The above theorem allows us to establish a sort of involutive solutions to AKNS soliton equations, which exhibits a kind of separation of variables for AKNS soliton equations. This is the following result.

Theorem 4.4 *The n-th AKNS soliton equation $u_{t_n}=K_n$ has the involutive solution with separated variables x, t_n*

$$\left\{\begin{array}{l} q=\sqrt{2}(< g_H^x g_{H_n}^{t_n} P_1(0,0), Bg_H^x g_{H_n}^{t_n} Q_2(0,0) > \\ \qquad\qquad + < g_H^x g_{H_n}^{t_n} P_2(0,0), Bg_H^x g_{H_n}^{t_n} Q_3(0,0) >), \\ r=\sqrt{2}(< g_H^x g_{H_n}^{t_n} P_2(0,0), Bg_H^x g_{H_n}^{t_n} Q_1(0,0) > \\ \qquad\qquad + < g_H^x g_{H_n}^{t_n} P_3(0,0), Bg_H^x g_{H_n}^{t_n} Q_2(0,0) >). \end{array}\right. \tag{4.12}$$

where g_G^y denotes the Hamiltonian phase flow of G with a variable y and $P_i(0,0)$ and $Q_i(0,0)$, $1\le i\le 3$, may be arbitrary initial value vectors.

Proof: Let

$$P_i(x,t_n)=g_H^x g_{H_n}^{t_n} P_i(0,0),\ Q_i(x,t_n)=g_H^x g_{H_n}^{t_n} Q_i(0,0),\ 1\le i\le 3.$$

Then $P_i(x,t_n)$ and $Q_i(x,t_n)$, $1\le i\le 3$, solve the nonlinearized spatial system (3.8) and the Hamiltonian system (4.11). However under the control of (3.8), (4.11) is equivalent to the nonlinearized temporal system (3.9). This shows that $P_i(x,t_n)$ and $Q_i(x,t_n)$ also solve (3.8) and (3.9), simultaneously. Therefore the compatibility condition of (3.8) and (3.9) is satisfied, i.e. (4.12) determines a solution to $u_{t_n}=K_n$. In addition, since $\{H,H_n\}=0$, the Hamiltonian phase flows g_H^x, $g_{H_n}^{t_n}$ may commute with each other. It follows that the resulting solution (4.12) is involutive. The proof is finished. ▮

5 Conclusions and remarks

We have introduced a three-by-three matrix spectral problem for the usual AKNS soliton hierarchy and proposed the corresponding Bargmann symmetry constraint on this AKNS hierarchy. Moreover we have exhibited an explicit Poisson algebra

$$\{\bar F_j,\ 1\le j\le N,\ F_m,\ m\ge 1\} \tag{5.1}$$

on the symplectic manifold $(\mathbb{R}^{6N},\omega^2)$ and further a binary nonlinearization procedure is manipulated along with a sort of involutive solutions to AKNS soliton equations. When $N=1,2$, we have proved the nonlinearized spatial system (3.8) is integrable in the Liouville sense, indeed. But we don't know if we can take out enough independent integrals of motion among the Poisson algebra (5.1) for a general integer N. We hope that this Poisson algebra suffices for proving complete integrability of the nonlinearized Lax systems.

It should be pointed out that the Neumann symmetry constraint and the higher order symmetry constraints

$$K_{-1} = J\sum_{j=1}^{N} E_j \frac{\delta \lambda_j}{\delta u},\ K_m = JG_m = J\sum_{j=1}^{N} E_j \frac{\delta \lambda_j}{\delta u},\ (m \geq 1), \tag{5.2}$$

may also be considered. These sorts of symmetry constraints are somewhat different from the Bargmann symmetry constraints because K_{-1} is a constant vector and the conserved covariants G_m, $m \geq 1$, involve the differential of the potential u with respect to the space variable x. In order to discuss them, we are required to introduce a new symplectic submanifold of the Euclidean spaces in the case of the Neumann constraint and new dependent variables, i.e. the so-called Jacobi-Ostrogradsky coordinates [16], in the case of higher order constraints. Similarly, we can consider the corresponding τ-symmetry (time first order dependent symmetry) constraints or more generally, time polynomial dependent symmetry constraints. Note that the similar Bargmann symmetry constraints have also been carefully analyzed for KP hierarchy [20] and the symmetries in the right hand side of the Bargmann symmetry constraints are sometimes called additional symmetries [10] and may be taken as sources of soliton equations [19].

We remark that the finite dimensional Hamiltonian systems generated by nonlinearization technique depend on the starting spectral problems. Therefore the same soliton equation may relate to different finite dimensional Hamiltonian systems once it possesses different Lax representations. AKNS soliton equations are exactly such examples. But we don't know if there exists an interrelation among the different finite dimensional Hamiltonian systems generated from the same soliton equation. In the binary nonlinearization procedure itself, there also exist some intriguing open problems. For example, why do the nonlinearized spatial system and the nonlinearized temporal systems for $n \geq 0$ under the control of the nonlinearized spatial system always possess Hamiltonian structures? We don't know either whether or not the nonlinearized temporal systems for $n \geq 0$ are themselves integrable soliton equations without the control of the nonlinearized spatial system. These problems are worth studying in order to enrich integrable structures of soliton equations.

Acknowledgments: One of the authors (W. X. Ma) would like to thank the Alexander von Humboldt Foundation for a research fellow award and the National Natural Science Foundation of China and the Shanghai Science and Technology Commission of China for financial support. He is also grateful to Drs. P. Zimmermann and G. Oevel for their helpful and stimulating discussions about MuPAD.

References

[1] M. R. Adams, J. Harnad and E. Previato, Isospectral Hamiltonian flows in finite and infinite dimensions, I. Generalized Moser systems and moment maps into loop algebras, Commun. Math. Phys. 117 (1988) 451–500.

[2] M. Antonowicz and S. Rauch Wojciechowski, Restricted flows of soliton hierarchies: coupled KdV and Harry Dym case, J. Phys. A: Math. Gen. 24 (1991) 5043–5061(1991).

[3] M. Antonowicz and S. Rauch Wojciechowski, How to construct finite dimensional bi-Hamiltonian systems from soliton equations: Jacobi integrable potentials, J. Math. Phys. 33 (1992) 2115–2125.

[4] V. I. Arnold, Mathematical Methods in Classical Mechanics (Springer-Verlag, New York, 1980).

[5] O. I. Bogoyavlenskii and S. P. Novikov, The relationship between Hamiltonian formalism of stationary and non stationary problems, Funct. Anal. Appl. 176 (1976) 8–11.

[6] M. Blaszak, Bi-Hamiltonian field Garnier system, Phys. Lett. A 174 (1993) 85–88; Miura map and bi-Hamiltonian formulation for restricted flows of the KdV hierarchy, J. Phys. A: Math. Gen. 26 (1993) 5985–5996.

[7] C. W. Cao, Nonlinearization of the Lax system for AKNS hierarchy, Sci. China A 33 (1990) 528–536.

[8] C. W. Cao and X. G. Geng, Classical integrable systems generated through nonlinearization of eigenvalue problems, in: Nonlinear Physics, eds. C. H. Gu, Y. S. Li and G. Z. Tu, (Springer Verlag, Berlin, 1990) 68–78.

[9] R. W. Carroll, Topics in Soliton Theory (North-Holland, Amsterdam, 1991).

[10] L. A. Dickey, On the constrained KP hierarchy, Lett. Math. Phys. 34 (1995) 379–384.

[11] H. Flaschka, Toward an algebro-geometrical interpretation of the Neumann system, Tohoku Math. J. 36 (1984) 407–426.

[12] A. S. Fokas and R. L. Anderson, On the use of isospectral eigenvalue problems for obtaining hereditary symmetries for Hamiltonian systems, J. Math. Phys. 23 (1982) 1066–1073.

[13] B. Fuchssteiner and A. S. Fokas, Symplectic structures, their Bäcklund transformations and hereditary symmetries, Physica D 4 (1981) 47–66.

[14] X. G. Geng, The Hamiltonian structure and new finite-dimensional integrable system associated with Harry-Dym type equations, Phys. Lett. A 194 (1994) 44–48.

[15] X. G. Geng and W. X. Ma, A generalized Kaup-Newell spectral problem, soliton equations and finite-dimensional integrable systems, Nuovo Cimento A 108 (1995) 477–486.

[16] M. de León and P. R. Rodrigues, Generalized Classical Mechanics and Field Theory (North-Holland, Amsterdam, 1985).

[17] W. X. Ma, New finite dimensional integrable systems by symmetry constraint of the KdV equations, J. Phys. Soc. Jpn. 64 (1995) 1085–1091; Symmetry constraint of MKdV equations by binary nonlinearization, to appear in Physica A.

[18] W. X. Ma and W. Strampp, An explicit symmetry constraint for Lax pairs and the adjoint Lax pairs of AKNS systems, Phys. Lett. A 185 (1994) 277–286.

[19] V. K. Mel'nikov, New method for deriving nonlinear integrable systems, J. Math. Phys. 31 (1990) 1106–1113.

[20] W. Oevel and W. Strampp, Constrained KP hierarchy and bi-Hamiltonian structures, Commun. Math. Phys. 157 (1993) 51–81.

[21] A. M. Perelomov, Integrable Systems of Classical Mechanics and Lie Algebras, Vol. I, (Birkhaeuser Verlag, Basel, 1990).

[22] O. Ragnisco and S. Rauch-Wojciechowski, Restricted flows of AKNS hierarchy, Inverse Problems 8 (1992) 245–262.

[23] R. J. Schilling, Generalizations of the Neumann system, a curve-theoretical approach–part I, Commun. Pure Appl. Math. 40 (1987) 455–522.

[24] G. Tondo, On the integrability of finite-dimensional systems related to soliton equations (preprint, Quaderno n. 340, Aprile 1995).

[25] G. Z. Tu, The trace identity, a powerful tool for constructing the Hamiltonian structure of integrable systems, J. Math. Phys. 30 (1989) 330–338.

[26] G. Z. Tu, R. I. Andrushkiw and X. C. Huang, A trace identity and its application to integrable systems of $1+2$ dimensions, J. Math. Phys. 32 (1991) 1900–1907.

[27] Y. B. Zeng, An approach to the deduction of the finite-dimensional integrability from the infinite-dimensional integrability, Phys. Lett. A 160 (1991) 541–547.

[28] Y. B. Zeng, New factorization of the Kaup-Newell hierarchy, Physica D 73 (1994) 171–188.