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Gauge transformations of constrained KP flows: new integrable hierarchies

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Abstract

Integrable systems in 1+1 dimensions arise from the KP hierarchy as symmetry reductions involving square eigenfunctions. Exploiting the residual gauge freedom in these constraints new integrable systems are derived. They include generalizations of the hierarchy of the Kundu-Eckhaus equation and higher order extensions of the Yajima-Oikawa and Melnikov hierarchies. Constrained modified KP flows yield further integrable equations such as the hierarchies of the derivative NLS equation, the Gerdjikov-Ivanov equation and the Chen-Lee-Liu equation .

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I Introduction

It is well established that the (scalar) KP hierarchy admits constraints [1-3] expressed through its symmetry $(\sum_i \Psi^{(i)}\Psi^{(i)*})_x$, where $\Psi^{(i)}$ and $\Psi^{(i)*}$ are eigenfunctions and adjoint eigenfunctions of the corresponding linear system. In particular, the constraints

$$u_x = (\sum_{i=1}^{m} \Psi^{(i)}\Psi^{(i)*})_x \ , \quad u_y = (\sum_{i=1}^{m} \Psi^{(i)}\Psi^{(i)*})_x \quad \text{and} \quad u_t = (\sum_{i=1}^{m} \Psi^{(i)}\Psi^{(i)*})_x \ , \tag{1.1}$$

are known to produce vector versions of the AKNS , the Yajima-Oikawa and the Melnikov hierarchy, respectively [3]. A simple but crucial observation, which inspired the present investigation, is that the constraints (1.1) do not fix uniquely the resulting equations obtained for $\Psi^{(i)}$ and $\Psi^{(i)*}$ and allows a residual gauge freedom

$$\tilde{\Psi}^{(i)} = e^{-\theta^{(i)}}\Psi^{(i)} \ , \quad \tilde{\Psi}^{(i)*} = \Psi^{(i)*}e^{\theta^{(i)}} \tag{1.2}$$

with arbitrary functions $\theta^{(i)}$. These fields may be used to generate gauge transformed integrable systems. We will use this gauge freedom to obtain new integrable systems which include generalizations of the Kundu-Eckhaus [4-6] hierarchy, a higher-order Yajima-Oikawa hierarchy as well as a higher-order Melnikov hierarchy. We note that some of these extensions exhibit an interesting breaking of global symmetry, while integrability is still preserved. A similar gauge freedom is found for the constrained modified KP hierarchy, investigated in the recent past [7, 8]. The gauge freedom is even richer in this case, and even the simplest constraint incorporates different equations such as the derivative NLS system [9], the Gerdjikov-Ivanov equation [10] and the Chen-Lee-Liu [11] system.

The organization of the paper is as follows. In Section 2 we introduce the main idea with a simple example leading to the hierarchy of the Kundu-Eckhaus equation. In Section 3 a consistent choice for the gauge function is proposed. This leads to a simple and general characterization of the integrable equations via an r-matrix formulation of their Lax representations. In Section 4 it is demonstrated that the simplest examples include extended Yajima-Oikawa and Melnikov systems. In Section 5 we briefly discuss alternative generalizations, while in Section 6 a similar construction is proposed for the constrained modified KP hierarchy.

II Constrained KP flows and the Kundu-Eckhaus hierarchy

The integrable KP hierarchy can be formulated through Zakharov-Shabat type linear systems [12]

$$\Psi_{t_l} = B_l\Psi \ , \quad \Psi^*_{t_l} = -B^*_l\Psi^* \ , \quad l = 1, 2, \ldots \tag{2.1}$$

with an infinite hierarchy of operators $B_l = (L^l)_+$, where the subscript + denotes the projection of the powers of the micro-differential Lax operator

$$L = \partial + u\,\partial^{-1} + u_2\,\partial^{-2} + u_3\,\partial^{-3} + \ldots \ , \quad \partial = \partial_x \tag{2.2}$$

onto its differential part. The leading coefficient u plays a distinguished role, since it satisfies the KP equation and its higher flows. The first of the operators B_l are computed as

$$B_1 = \partial\,, \quad B_2 = \partial^2 + 2u\,, \quad B_3 = \partial^3 + 3u\partial + 3u_2 + 3u_x\,. \tag{2.3}$$

The Lax equations

$$L_{t_l} = [B_l, L]\;, \quad l = 1, 2, \ldots \tag{2.4}$$

imply the compatibility conditions

$$B_{l,t_k} - B_{k,t_l} + [B_l, B_k] = 0 \tag{2.5}$$

of (2.1). They yield the field equations along with differential relations among the fields u_i such as

$$\begin{aligned} u_2 &= -\tfrac{1}{2}u_x + \tfrac{1}{2}\partial^{-1}u_y\;, \\ u_3 &= \tfrac{1}{4}u_{xx} - \tfrac{1}{2}u_y - \tfrac{1}{2}u^2 + \tfrac{1}{4}\partial^{-2}u_{yy}\;. \end{aligned} \tag{2.6}$$

Consequently all equations of this hierarchy can be expressed through the single field u.

We will refer to solutions of (2.1) as (adjoint) eigenfunctions, although in the following no spectral problem $L\Psi = \lambda\Psi$, $L^*\Psi^* = \lambda\Psi^*$ will be assumed. This corresponds to the Inverse Scattering Transform of the KP hierarchy, in which the evolution equation $\Psi_y = B_2\Psi$ associated with $y = t_2$ actually is the "spatial" part of the scattering problem. Any (formal) spectral equation would only introduce some (asymptotic) dependence on a spectral parameter, which bears no relevance for the following.

It has been observed that the product $\Psi\Psi^*$ of an eigenfunction and an adjoint eigenfunction represents a conserved covariant of the KP hierarchy. Hence, one may impose constraints on the KP flows by expressing the dynamical field u through $\Psi\Psi^*$ [1-3]. In terms of the Lax operator (2.2) these constraints are characterized by the requirement that the negative differential orders of a power L^k have the specific form

$$(L^k)_- = \sum_{i=1}^{m} \Psi^{(i)}\partial^{-1}\Psi^{(i)*}\;, \tag{2.7}$$

where m pairs of (adjoint) eigenfunctions are considered. For given k this constraint leaves the coefficients $u, u_2, .., u_{k-1}$ in (2.2) and $\Psi^{(i)}$, $\Psi^{(i)*}$ as independent fields, whereas $u_k, u_{k+1}, ..$ become differential expressions of these functions. One may replace L by the new Lax operator

$$L' = L^k = \partial^k + k\,u\,\partial^{k-2} + \sum_{j=0}^{k-3} U_j\,\partial^j + \sum_{i=1}^{m} \Psi^{(i)}\partial^{-1}\Psi^{(i)*}\;, \tag{2.8}$$

where $U_0, .., U_{k-3}$ are differential expressions of $u, u_2, .., u_{k-1}$. Thus the fields $u, U_0, .., U_{k-3}$ may be regarded as new independent fields related to $u, u_2, .., u_{k-1}$ by a coordinate transformation. It is readily verified that the Lax equations (2.4) subject to the constraint

(2.7) automatically imply that $\Psi^{(i)}$ and $\Psi^{(i)*}$ are (adjoint) eigenfunctions satisfying (2.1). We note that these constraints may indeed be regarded as symmetry reductions of the KP-hierarchy, since

$$u_{t_k} = res(L)_{t_k} = res([(L^k)_+, L]) = res([L_+, L^k]) = res(L^k)_x \ , \tag{2.9}$$

so that (2.7) implies the relation

$$u_{t_k} = (\sum_{i=1}^{m} \Psi^{(i)}\Psi^{(i)*})_x \tag{2.10}$$

between the kth flow of the KP hierarchy and the symmetry generated by square eigenfunctions. The simplest of these constraints $k=1$, $m=1$ is given by

$$L = \partial + \Psi\partial^{-1}\Psi^* = \partial + \Psi\Psi^*\,\partial^{-1} - \Psi\Psi^*_x\,\partial^{-2} + \Psi\Psi^*_{xx}\,\partial^{-3} \mp \dots \ , \tag{2.11}$$

i.e.

$$u = \Psi\Psi^* \ , \quad u_2 = -\Psi\Psi^*_x \ , \quad u_3 = \Psi\Psi^*_{xx} \ , \quad \dots \ . \tag{2.12}$$

In this case (2.1)/(2.4) yields the AKNS hierarchy for Ψ, Ψ^*, while u solves the KP equation and its higher flows.

We now want to motivate the considerations of the next sections using the AKNS constraint (2.12). We observe that in the factorization $u = \Psi\Psi^*$ of the KP hierarchy there still remains a gauge freedom. One may introduce gauge transformed (adjoint) eigenfunctions

$$\tilde{\Psi} = h^{-1}\Psi \ , \quad \tilde{\Psi}^* = \Psi^* h \tag{2.13}$$

with an arbitrary function h. It is noted that under such a change the KP solution u remains the same, while the corresponding AKNS hierarchy is transformed into

$$\tilde{\Psi}_{t_l} = \tilde{B}_l\tilde{\Psi} \ , \quad \tilde{\Psi}^*_{t_l} = -\tilde{B}^*_l\tilde{\Psi}^* \ , \quad l = 1, 2, \dots \tag{2.14}$$

with the new differential operators

$$\tilde{B}_l = h^{-1} \circ B_l \circ h - h^{-1}h_{t_l} \ . \tag{2.15}$$

We introduce the gauge field in the form $h = e^{\theta}$, which simplifies the structure of the transformed operators $\tilde{B}_l$. For example, with $t_2 = y$ one obtains

$$\tilde{B}_2 = B_2 + 2\theta_x\partial - \theta_y + \theta_x^2 + \theta_{xx} \tag{2.16}$$

and the corresponding linear equations

$$\begin{aligned} \tilde{\Psi}_y &= \tilde{\Psi}_{xx} + 2\theta_x\tilde{\Psi}_x + (2u - \theta_y + \theta_x^2 + \theta_{xx})\tilde{\Psi} \ , \\ \tilde{\Psi}^*_y &= -\tilde{\Psi}^*_{xx} + 2\theta_x\tilde{\Psi}^*_x - (2u - \theta_y + \theta_x^2 - \theta_{xx})\tilde{\Psi}^* \ . \end{aligned} \tag{2.17}$$

We observe that one may choose the gauge field θ as the potential defined by

$$\begin{aligned} \theta_x &= \tilde{\Psi}\tilde{\Psi}^* = u \ , \\ \theta_y &= \tilde{\Psi}_x\tilde{\Psi}^* - \tilde{\Psi}\tilde{\Psi}^*_x + 2(\tilde{\Psi}\tilde{\Psi}^*)^2 \ . \end{aligned} \tag{2.18}$$

The consistency $\theta_{xy} = \theta_{yx}$ may be checked directly from the equations (2.17) and (2.18). Elimination of θ and u finally yields the following gauge transformed AKNS system

$$\begin{aligned} \tilde{\Psi}_y &= \tilde{\Psi}_{xx} + \Big(2\tilde{\Psi}\tilde{\Psi}^* + 2\,(\tilde{\Psi}\tilde{\Psi}^*)_x - (\tilde{\Psi}\tilde{\Psi}^*)^2 \Big)\,\tilde{\Psi} \ , \\ \tilde{\Psi}_y &= -\tilde{\Psi}_{xx} - \Big(2\,\tilde{\Psi}\tilde{\Psi}^* - 2\,(\tilde{\Psi}\tilde{\Psi}^*)_x - (\tilde{\Psi}\tilde{\Psi}^*)^2 \Big)\,\tilde{\Psi}^* \ , \end{aligned} \tag{2.19}$$

which is recognized as the Kundu-Eckhaus equation [4, 5]. At this stage the definition of θ via (2.18) seems ad hoc. We will show in the next section that there is a systematic construction of the gauge field in terms of the conserved densities of the KP hierarchy, i.e. θ may be chosen to satisfy the potential KP hierarchy. Adjoining a suitable compatible time evolution with respect to the next time $t = t_3$ to the definition (2.18), one obtains the next higher flow of the Kundu-Eckhaus hierarchy. Any simultaneous solution of this system and (2.19), each of them being a system in 1+1 dimensions, gives rise to a KP solution $u = \Psi\Psi^* = \tilde{\Psi}\tilde{\Psi}^*$.

III Consistent choice of the gauge function

The transformation

$$\Psi \mapsto \tilde{\Psi} = e^{-\theta}\Psi \ , \quad \Psi^* \mapsto \tilde{\Psi}^* = \Psi^* e^{\theta} \tag{3.1}$$

of the (adjoint) eigenfunctions for the Lax operator (2.2) produces (adjoint) eigenfunctions of the gauge transformed Lax operator

$$\tilde{L} = e^{-\theta} \circ L \circ e^{\theta} = \partial + \theta_x + u(\partial + \theta_x)^{-1} + u_2(\partial + \theta_x)^{-2} + \dots \ , \tag{3.2}$$

where we note

$$e^{-\theta} \circ \partial^i \circ e^{\theta} = (e^{-\theta} \circ \partial \circ e^{\theta})^i = (\partial + \theta_x)^i \tag{3.3}$$

for arbitrary powers i. The Lax equations (2.4) are mapped to corresponding equations for $\tilde{L}$ with $\tilde{B}_l$ given by (2.15), i.e.

$$\tilde{B}_l = e^{-\theta} \circ (L^l)_+ \circ e^{\theta} - \theta_{t_l} = (\tilde{L}^l)_+ - \theta_{t_l} \ . \tag{3.4}$$

The choice of the gauge function θ corresponding to the example of the last section is the following. One defines θ as the potential characterized by

$$\theta_{t_l} = \epsilon\ res(L^l) = \epsilon\ res(\tilde{L}^l) \ , \quad l = 1, 2, \dots \ , \tag{3.5}$$

where res is the usual residue of a micro-differential operator, i.e. the coefficient of ∂^{-1}. The arbitrary constant ϵ is introduced as a deformation parameter.

We remark that any choice of θ given by an arbitrary function of the KP fields, i.e. the coefficients u_i of L in (2.2), gives rise to a closed system of equations for the gauge transformed fields. However, in this case the transformation represents a mere "change of variables" $u_i \mapsto F_i(u_1, u_2, \ldots)$ with some differential expressions F_i. With the choice (3.5) θ is a non-trivial potential, still preserving the local character of the reductions to be considered in the following.

The compatibility $\theta_{t_m t_n} = \theta_{t_n t_m}$ is easily verified, since the Lax equations (2.4) imply

$$\theta_{t_m t_n} = \epsilon\, res([(L^n)_+, L^m]) = \epsilon\, res(L^{m+n} - (L^m)_- L^n - (L^n)_- L^m)\ , \tag{3.6}$$

which is clearly symmetric in m and n. The defining equations (3.5) determine θ up to an integration constant. This does not contribute to the gauge transformation (3.2), and consequently $\tilde{L}$ is characterized in terms of derivatives of θ. We note that the residues represent conserved densities of the KP hierarchy, since

$$res(L^l)_{t_n} = res([(L^n)_+, L^l]) \tag{3.7}$$

is a perfect x-derivative. For $l = 1$ equation (3.5) yields $\theta_x = \epsilon\, u$, so that θ is a solution of the potential KP hierarchy. Consequently, θ is the highest non-trivial coefficient in the dressing operator $W = 1 + \theta\, \partial^{-1} + (\ldots)\partial^{-2} + \cdots$, which generates $L = W^{-1}\partial W$ by dressing the bare operator ∂. Also, we note the link $\theta = \tau_x/\tau$ to the τ function of the KP hierarchy.

One finally obtains the deformed Lax operator

$$\begin{aligned} \tilde{L} &= \partial + \epsilon\, u + u\,(\partial + \epsilon\, u)^{-1} + u_2\,(\partial + \epsilon\, u)^{-2} + \ldots \\ &= \partial + \epsilon\, u + u\,\partial^{-1} + (u_2 - \epsilon\, u^2)\,\partial^{-2} + \ldots\ . \end{aligned} \tag{3.8}$$

With (3.4) and (3.5) the new Lax equations are given by

$$\tilde{L}_{t_l} = [\tilde{B}_l, \tilde{L}] = [r(\tilde{L}^l), \tilde{L}] \tag{3.9}$$

with

$$r(A) = A_+ - \epsilon\, res(A) - \frac{1}{2}\, A = \frac{1}{2}\,(A_+ - A_-) - \epsilon\, res(A)\ . \tag{3.10}$$

We note that this map satisfies the modified Yang-Baxter equation [13]

$$[r(A), r(B)] + \frac{1}{4}\,[A, B] = r([r(A), B] + [A, r(B)]) \tag{3.11}$$

for all micro-differential operators A, B and any $\epsilon \in \mathbb{R}$, so that (3.10) yields the r-matrix of the flows (3.9).

Up to this point one has not gained any new results from the gauge transformation of the KP hierarchy. However, we now look for reductions to 1+1 dimensions, where the new formulation yields results which are different from the standard reductions. The k-constraint (2.7) yields

$$(\tilde{L}^k)_- = e^{-\theta} \circ (L^k)_- \circ e^{\theta} = \sum_{i=1}^{m} e^{-\theta} \Psi^{(i)} \partial^{-1} \Psi^{(i)*} e^{\theta} = \sum_{i=1}^{m} \tilde{\Psi}^{(i)} \partial^{-1} \tilde{\Psi}^{(i)*} \ . \tag{3.12}$$

One may replace the Lax operator (3.8) by its kth power $\tilde{L}' = \tilde{L}^k$ and obtains the Lax representation

$$\tilde{L}'_{t_l} = [\tilde{B}_l, \tilde{L}'] \ , \quad \tilde{B}_l = (\tilde{L}'^{l/k})_+ - \epsilon\, res(\tilde{L}'^{l/k}) \ , \quad l = 1, 2, \ldots \tag{3.13}$$

for Lax operators of the form

$$\tilde{L}' = (\partial + \epsilon\, u)^k + k\, u\, (\partial + \epsilon\, u)^{k-2} + \sum_{j=0}^{k-3} U_j (\partial + \epsilon\, u)^j + \sum_{i=1}^{m} \tilde{\Psi}^{(i)} \partial^{-1} \tilde{\Psi}^{(i)*} \ , \tag{3.14}$$

which are deformations of the operators (2.8). By construction the negative differential orders of (3.13) are compatible with the assumption that $\tilde{\Psi}^{(i)}$ and $\tilde{\Psi}^{(i)*}$ are (adjoint) eigenfunctions satisfying

$$\tilde{\Psi}^{(i)}_{t_l} = \tilde{B}_l \tilde{\Psi}^{(i)} \ , \quad \tilde{\Psi}^{(i)*}_{t_l} = -\tilde{B}^*_l \tilde{\Psi}^{(i)*} \ . \tag{3.15}$$

We finally note that the deformation process induced by the parameter ϵ may also be regarded as a Miura type transformation between the equations given by (2.1) and (3.15), respectively. Elimination of θ from (3.1) yields the Bäcklund relations

$$\Big(ln\Big(\frac{\Psi^{(i)}}{\tilde{\Psi}^{(i)}}\Big)\Big)_{t_l} = \Big(ln\Big(\frac{\tilde{\Psi}^{(i)*}}{\Psi^{(i)*}}\Big)\Big)_{t_l} = \theta_{t_l} = \epsilon\, res(\tilde{L}^l) \ , \tag{3.16}$$

with the spatial part

$$\Big(ln\Big(\frac{\Psi^{(i)}}{\tilde{\Psi}^{(i)}}\Big)\Big)_{x} = \Big(ln\Big(\frac{\tilde{\Psi}^{(i)*}}{\Psi^{(i)*}}\Big)\Big)_{x} = \theta_x = \epsilon\, u \ . \tag{3.17}$$

They provide the transformation from $\Psi^{(i)}$, $\Psi^{(i)*}$ associated with $\epsilon = 0$ to the gauge transformed (adjoint) eigenfunctions $\tilde{\Psi}^{(i)}$, $\tilde{\Psi}^{(i)*}$ associated with arbitrary ϵ.

IV Examples: extended AKNS, Yajima-Oikawa and Melnikov hierarchies

Depending on the choice of the integer k we find the following examples, where we use the notation $t_1 = x$, $t_2 = y$ and $t_3 = t$:

Example 1: For $k=1$ one finds

$$\begin{aligned}\theta_x &= \epsilon\, res(\tilde{L}) = \epsilon\, u = \epsilon \sum_{i=1}^{m} \tilde{\Psi}^{(i)}\tilde{\Psi}^{(i)*} , \\ \theta_y &= \epsilon\, res(\tilde{L}^2) = \epsilon \sum_{i=1}^{m} (\tilde{\Psi}_x^{(i)}\tilde{\Psi}^{(i)*} - \tilde{\Psi}^{(i)}\tilde{\Psi}_x^{(i)*}) + 2\,\epsilon^2\, u^2 ,\end{aligned} \tag{4.1}$$

for the Lax operator

$$\tilde{L} = \partial + \epsilon\, u + \sum_{i=1}^{m} \tilde{\Psi}^{(i)}\partial^{-1}\tilde{\Psi}^{(i)*} = \partial + \sum_{i=1}^{m} \tilde{\Psi}^{(i)}\ (\partial^{-1} + \epsilon)\ \tilde{\Psi}^{(i)*} . \tag{4.2}$$

The basic equation of the corresponding vector Kundu-Eckhaus hierarchy for the fields $\tilde{\Psi}^{(1)}, .., \tilde{\Psi}^{(m)*}$ is

$$\begin{aligned}\tilde{\Psi}_y^{(i)} &= \tilde{\Psi}_{xx}^{(i)} + 2\,\epsilon\, u\, \tilde{\Psi}_x^{(i)} + \Big(\ 2\,u - \epsilon^2\, u^2 + 2\,\epsilon \sum_{j=1}^{m} \tilde{\Psi}^{(j)}\tilde{\Psi}_x^{(j)*}\Big)\ \tilde{\Psi}^{(i)} , \\ \tilde{\Psi}_y^{(i)*} &= -\tilde{\Psi}_{xx}^{(i)*} + 2\,\epsilon\, u\, \tilde{\Psi}_x^{(i)*} + \Big(-2\,u + \epsilon^2\, u + 2\,\epsilon \sum_{j=1}^{m} \tilde{\Psi}_x^{(j)}\tilde{\Psi}^{(j)*}\Big)\ \tilde{\Psi}^{(i)*} ,\end{aligned} \tag{4.3}$$

where one has to insert the constraint $u = \sum_j \tilde{\Psi}^{(j)}\tilde{\Psi}^{(j)*}$ of the KP hierarchy. This equation reduces to (2.19) for a single pair of (adjoint) eigenfunctions and $\epsilon = 1$. For $\epsilon = 0$ one obtains the usual AKNS system.

Example 2: For $k=2$, i.e for the constraint

$$u_y = (\sum_{i=1}^{m} \tilde{\Psi}^{(i)}\tilde{\Psi}^{(i)*})_x \tag{4.4}$$

of the KP hierarchy one obtains

$$\theta_x = \epsilon\ res(\tilde{L}) = \epsilon\, u\ , \quad \theta_y = \epsilon\ res(\tilde{L}^2) = \epsilon \sum_{i=1}^{m} \tilde{\Psi}^{(i)}\tilde{\Psi}^{(i)*} . \tag{4.5}$$

The first flow of the corresponding hierarchy of equations (3.13) for $u, \tilde{\Psi}^{(i)}, \tilde{\Psi}^{(i)*}$ associated with the Lax operator

$$\tilde{L}' = \tilde{L}^2 = (\partial + \epsilon\, u)^2 + 2\,u + \sum_{i=1}^{m} \tilde{\Psi}^{(i)}\partial^{-1}\tilde{\Psi}^{(i)*} \tag{4.6}$$

is given by

$$\begin{aligned}\tilde{\Psi}_y^{(i)} &= \tilde{\Psi}_{xx}^{(i)} + 2\,\epsilon\, u\, \tilde{\Psi}_x^{(i)} + \Big(\ 2\,u + \epsilon\, u_x + \epsilon^2\, u^2 - \epsilon \sum_{j=1}^{m} \tilde{\Psi}^{(j)}\tilde{\Psi}^{(j)*}\Big)\ \tilde{\Psi}^{(i)} , \\ \tilde{\Psi}_y^{(i)*} &= -\tilde{\Psi}_{xx}^{(i)*} + 2\,\epsilon\, u\, \tilde{\Psi}_x^{(i)*} + \Big(-2\,u + \epsilon\, u_x - \epsilon^2\, u^2 + \epsilon \sum_{j=1}^{m} \tilde{\Psi}^{(j)}\tilde{\Psi}^{(j)*}\Big)\ \tilde{\Psi}^{(i)*} ,\end{aligned} \tag{4.7}$$

complemented by (4.4). For $\epsilon = 0$ one obtains the Yajima-Oikawa system.

Example 3: The next higher constraint (2.7) with $k = 3$ implies

$$u_t = (\sum_{i=1}^{m} \tilde{\Psi}^{(i)} \tilde{\Psi}^{(i)*})_x \ , \tag{4.8}$$

so that the KP equation becomes

$$u_y = v_x \ , \qquad v_y = -\frac{1}{3} u_{xxx} - 4 u u_x + \frac{4}{3} \Big(\sum_{i=1}^{m} \tilde{\Psi}^{(i)} \tilde{\Psi}^{(i)*} \Big)_x \ , \tag{4.9}$$

where we have put $u_2 = (v - u_x)/2$ in (2.2). One finds

$$\begin{aligned} &\theta_x = \epsilon \, res(\tilde{L}) = \epsilon \, u \ , \quad \theta_y = \epsilon \, res(\tilde{L}^2) = \epsilon \, v \ , \\ &\theta_t = \epsilon \, res(\tilde{L}^3) = \epsilon \sum_{i=1}^{m} \tilde{\Psi}^{(i)} \tilde{\Psi}^{(i)*} \ , \end{aligned} \tag{4.10}$$

and the first equations of the gauge transformed hierarchy for $u, v, \tilde{\Psi}^{(i)}, \tilde{\Psi}^{(i)*}$ are given by

$$\begin{aligned} \tilde{\Psi}^{(i)}_y &= \tilde{\Psi}^{(i)}_{xx} + 2 \epsilon \, u \, \tilde{\Psi}^{(i)}_x + \Big(2 \, u + \epsilon \, u_x - \epsilon v + \epsilon^2 \, u^2 \Big) \, \tilde{\Psi}^{(i)} \ , \\ \tilde{\Psi}^{(i)*}_y &= -\tilde{\Psi}^{(i)*}_{xx} + 2 \epsilon \, u \, \tilde{\Psi}^{(i)*}_x + \Big(-2 \, u + \epsilon \, u_x + \epsilon \, v - \epsilon^2 \, u^2 \Big) \, \tilde{\Psi}^{(i)*} \ , \end{aligned} \tag{4.11}$$

together with the equations (4.9). The next higher flow is computed as

$$\begin{aligned} \tilde{\Psi}^{(i)}_t &= \tilde{\Psi}^{(i)}_{xxx} + 3 \epsilon \, u \, \tilde{\Psi}^{(i)}_{xx} + 3 \, (u + \epsilon \, u_x + \epsilon^2 \, u^2) \, \tilde{\Psi}^{(i)}_x \\ &\quad + \Big(\frac{3}{2} \, u_x + \frac{3}{2} \, v + \epsilon \, u_{xx} + 3 \, \epsilon \, u^2 + 3 \, \epsilon^2 \, u u_x + \epsilon^3 \, u^3 - \epsilon \sum_{j=1}^{m} \tilde{\Psi}^{(j)} \tilde{\Psi}^{(j)*} \Big) \tilde{\Psi}^{(i)} \\ \tilde{\Psi}^{(i)*}_t &= \tilde{\Psi}^{(i)*}_{xxx} - 3 \epsilon \, u \, \tilde{\Psi}^{(i)*}_{xx} + 3 \, (u - \epsilon \, u_x + \epsilon^2 \, u^2) \, \tilde{\Psi}^{(i)*}_x \\ &\quad + \Big(\frac{3}{2} \, u_x - \frac{3}{2} \, v - \epsilon \, u_{xx} - 3 \, \epsilon \, u^2 + 3 \, \epsilon^2 \, u u_x - \epsilon^3 \, u^3 + \epsilon \sum_{j=1}^{m} \tilde{\Psi}^{(j)} \tilde{\Psi}^{(j)*} \Big) \tilde{\Psi}^{(i)*} \end{aligned} \tag{4.12}$$

along with the t-evolution equations given by the constraint (4.8) and

$$v_t = \sum_{j=1}^{m} \Big(\tilde{\Psi}^{(j)}_x \tilde{\Psi}^{(j)*} - \tilde{\Psi}^{(j)} \tilde{\Psi}^{(j)*}_x + 2 \, \epsilon \, u \, \tilde{\Psi}^{(j)} \tilde{\Psi}^{(j)*} \Big)_x \ . \tag{4.13}$$

This last equation follows from (4.11) with

$$v_t = res(\tilde{L}^2)_{t_3} = res(\tilde{L}^3)_{t_2} = \Big(\sum_{i=1}^{m} \tilde{\Psi}^{(i)} \tilde{\Psi}^{(i)*} \Big)_y \ . \tag{4.14}$$

The associated Lax operator is

$$\tilde{L}' = \tilde{L}^3 = (\partial + \epsilon \, u)^3 + \frac{3}{2} \, u \, (\partial + \epsilon \, u) + \frac{3}{2} \, (\partial + \epsilon \, u) \, u + \frac{3}{2} v + \sum_{i=1}^{m} \tilde{\Psi}^{(i)} \partial^{-1} \tilde{\Psi}^{(i)*} \ . \tag{4.15}$$

For $\epsilon = 0$ one obtains the standard Melnikov hierarchy.

V Multicomponent generalizations

It was shown that the vector AKNS equation (4.3) with $\epsilon = 0$ is gauge transformed into the vector Kundu-Eckhaus equation (4.3) with arbitrary ϵ. Apparently the invariance of the original AKNS system under the $SU(m)$ transformations

$$(\tilde{\Psi}^{(i)}, .., \tilde{\Psi}^{(m)}) \mapsto (\tilde{\Psi}^{(i)}, .., \tilde{\Psi}^{(m)})\, U \ , \quad (\tilde{\Psi}^{(i)*}, .., \tilde{\Psi}^{(m)*})^{\dagger} \mapsto U^{\dagger}\, (\tilde{\Psi}^{(i)*}, .., \tilde{\Psi}^{(m)*})^{\dagger} \quad (5.1)$$

with $UU^{\dagger} = I$ is preserved in this deformation. We note that this symmetry is easily broken, while still preserving integrability, when more general gauge transformations (1.2) are considered, where $\theta^{(i)}$ is different for each pair of (adjoint) eigenfunctions. The gauge fields may be chosen consistently as the potentials defined by

$$\theta^{(i)}_{t_l} = \epsilon^{(i)}\ res(\partial^{-1}\Psi^{(i)*}L^{l}\Psi^{(i)}\partial^{-1}) \ , \quad l = 1, 2, \ldots \quad (5.2)$$

implying

$$\theta^{(i)}_{x} = \epsilon^{(i)}\ \Psi^{(i)*}\Psi^{(i)} \quad (5.3)$$

for $l = 1$. Here not only the potentials, but also the deformation parameters $\epsilon^{(i)}$ are chosen independently for each pair. The proof of the integrability condition $\theta^{(i)}_{t_l t_m} = \theta^{(i)}_{t_m t_l}$ may be found in [14] (Lemma 1) or [8].

For the simplest constraint $k = 1$, i.e.

$$u = \sum_{i=1}^{m} \Psi^{(i)}\Psi^{(i)*} = \sum_{i=1}^{m} \tilde{\Psi}^{(i)}\tilde{\Psi}^{(i)*} \ , \quad L = \partial + \sum_{i=1}^{m} \Psi^{(i)}\,\partial^{-1}\,\Psi^{(i)*} \ , \quad (5.4)$$

one obtains

$$\begin{aligned}
\theta^{(i)}_{x} &= \epsilon^{(i)}\,\Psi^{(i)}\Psi^{(i)*} = \epsilon^{(i)}\,\tilde{\Psi}^{(i)}\tilde{\Psi}^{(i)*} \ , \\
\theta^{(i)}_{y} &= \epsilon^{(i)}\,(\,\Psi^{(i)}_{x}\Psi^{(i)*} - \Psi^{(i)}\Psi^{(i)*}_{x}\,) \\
&= \epsilon^{(i)}\,(\,\tilde{\Psi}^{(i)}_{x}\tilde{\Psi}^{(i)*} - \tilde{\Psi}^{(i)}\tilde{\Psi}^{(i)*}_{x}\,) + 2\,\epsilon^{(i)2}\,(\tilde{\Psi}^{(i)}\tilde{\Psi}^{(i)*})^{2} \ ,
\end{aligned} \quad (5.5)$$

and the transformations (1.2) map (2.1) into the following multicomponent generalization of the Kundu-Eckhaus equation

$$\begin{aligned}
\tilde{\Psi}^{(i)}_{y} &= \tilde{\Psi}^{(i)}_{xx} + \Big(\ 2\,u + 2\,\epsilon^{(i)}\,(\tilde{\Psi}^{(i)}\tilde{\Psi}^{(i)*})_{x} - \epsilon^{(i)2}\,(\tilde{\Psi}^{(i)}\tilde{\Psi}^{(i)*})^{2}\,\Big)\tilde{\Psi}^{(i)} \ , \\
\tilde{\Psi}^{(i)*}_{y} &= -\tilde{\Psi}^{(i)*}_{xx} + \Big(-\,2\,u + 2\,\epsilon^{(i)}\,(\tilde{\Psi}^{(i)}\tilde{\Psi}^{(i)*})_{x} + \epsilon^{(i)2}\,(\tilde{\Psi}^{(i)}\tilde{\Psi}^{(i)*})^{2}\,\Big)\tilde{\Psi}^{(i)*} \ ,
\end{aligned} \quad (5.6)$$

where u has to be inserted from (5.4). It is significant to observe that the original global $SU(m)$ symmetry has been broken down to just $U(1)$ due to the presence of anisotropic extensions. We remark that the field $\theta = \epsilon \sum_i \theta^{(i)}/\epsilon^{(i)}$ is the potential of the last section satisfying $\theta_{t_l} = \epsilon\, res(L^l)$. This is easily seen from integrating the relation

$$\begin{aligned}
\theta_{t_l x} &= \epsilon \sum_{i=1}^{m} res(\partial^{-1}\Psi^{(i)*}L^l\Psi^{(i)}\partial^{-1})_x &=& \epsilon \sum_{i=1}^{m} res([\,\partial\,,\,\partial^{-1}\Psi^{(i)*}L^l\Psi^{(i)}\partial^{-1}\,]) \\
&= \epsilon\, res([\,L^l\,,\,\sum_{i=1}^{m}\Psi^{(i)}\partial^{-1}\Psi^{(i)*}\,]) &=& \epsilon\, res([\,L^l\,,\,L_-\,]) \\
&= \epsilon\, res([\,L_+\,,\,L^l\,]) \;=\; \epsilon\, res([\,\partial\,,\,L^l\,]) &=& \epsilon\, res(L^l)_x\ .
\end{aligned} \tag{5.7}$$

Hence, for a single pair of (adjoint) eigenfunctions $m = 1$, it is no surprise that the multicomponent extensions (4.3) and (5.6) of the Kundu-Eckhaus equation coincide.

VI Gauge transformations of the constrained modified KP hierarchy

The modified KP hierarchy arises from the Lax representation ([15, 8])

$$L_{t_l} = [B_l, L]\ , \quad B_l = (L^l)_{\geq 1}\ , \quad l = 1, 2, ... \tag{6.1}$$

where $(.)_{\geq 1}$ denotes the projection to differential orders strictly larger than zero. The Lax operator is given by

$$L = \partial + v + u\,\partial^{-1} + v_2\,\partial^{-2} + v_3\,\partial^{-3} + \cdots\ , \tag{6.2}$$

where v solves the modified KP equation and u solves the KP equation. The formal adjoints of the linear problems $\Psi_{t_l} = B_l\Psi$ can be integrated, so that we regard

$$\Psi_{t_l} = B_l\Psi\ , \quad \Psi^*_{t_l} = -\partial^{-1}B_l^*\,\partial\,\Psi^* \tag{6.3}$$

as the associated linear (adjoint) problems. With $t_2 = y$ the first non-trivial linear problem is given by

$$\begin{aligned}
\Psi_y^{(i)} &= \Psi_{xx}^{(i)} + 2\,v\,\Psi_x^{(i)}\ , \\
\Psi_y^{(i)*} &= -\Psi_{xx}^{(i)*} + 2\,v\,\Psi_x^{(i)*}\ .
\end{aligned} \tag{6.4}$$

Our aim is to demonstrate that the gauge transformation formulated for the constrained KP system is applicable also to this case. As before, we consider the gauge transformation

$$\tilde{\Psi}^{(i)} = e^{-\theta}\,\Psi^{(i)}\ , \quad \tilde{\Psi}^{(i)*} = \Psi^{(i)*}\,e^{\theta}\ , \tag{6.5}$$

which maps (6.4) into

$$\begin{aligned}
\tilde{\Psi}_y^{(i)} &= \tilde{\Psi}_{xx}^{(i)} + 2\,(v+\theta_x)\,\tilde{\Psi}_x^{(i)} + (\theta_{xx} + \theta_x^2 + 2\,v\,\theta_x - \theta_y)\,\tilde{\Psi}^{(i)}\ , \\
\tilde{\Psi}_y^{(i)*} &= -\tilde{\Psi}_{xx}^{(i)*} + 2\,(v+\theta_x)\,\tilde{\Psi}_x^{(i)*} + (\theta_{xx} - \theta_x^2 - 2\,v\,\theta_x + \theta_y)\,\tilde{\Psi}^{(i)*}\ .
\end{aligned} \tag{6.6}$$

A consistent choice of the gauge field θ is given with the potential defined by

$$\theta_{t_l} = \epsilon_1 \; res(\, L^l \,) + \epsilon_2 \; res(\, L^l \, \partial^{-1} \,) \tag{6.7}$$

with two arbitrary deformation parameters $\epsilon_{1,2}$. The compatibility $\theta_{t_n t_m} = \theta_{t_m t_n}$ is verified from the relation

$$\begin{aligned} \theta_{t_m t_n} &= res(\, [\, (L^n)_{\geq 1} \, , \, L^m \,] \, (\epsilon_1 + \epsilon_2 \, \partial^{-1}) \,) \\ &= res(\, \Big(\, L^{m+n} - (L^m)_{<1} L^n - (L^n)_{<1} L^m + (L^n)_{<1} (L^m)_{<1} \, \Big) \; (\epsilon_1 + \epsilon_2 \, \partial^{-1}) \,) \; , \end{aligned} \tag{6.8}$$

where $(.)_{<1}$ denotes the projection to non-positive differential orders. With the operator identities

$$res((L^n)_{<1} (L^m)_{<1}) = res(L^n \partial^{-1}) \; res(L^m) + res(L^n) \; res(L^m \partial^{-1}) \tag{6.9}$$

and

$$res((L^n)_{<1} (L^m)_{<1} \partial^{-1}) = res(L^n \partial^{-1}) \; res(L^m \partial^{-1}) \tag{6.10}$$

this is clearly symmetric in n and m. For $l = 1, 2$ the definition (6.7) yields

$$\begin{aligned} \theta_x &= \epsilon_1 \; u + \epsilon_2 \; v \; , \\ \theta_y &= \epsilon_1 \; (u_x + 2 \, uv + 2 \, v_2) + \epsilon_2 \; (2 \, u + v_x + v^2) \; , \end{aligned} \tag{6.11}$$

which after insertion into (6.6) leads to

$$\begin{aligned} \tilde{\Psi}^{(i)}_y &= \quad \tilde{\Psi}^{(i)}_{xx} \; + 2 \Big(\, v + \epsilon_1 \, u + \epsilon_2 \, v \Big) \, \tilde{\Psi}^{(i)}_x \\ &\qquad + \Big(- 2 \, \epsilon_1 \, v_2 + \epsilon_2 \, (-2 \, u + v^2) + (\epsilon_1 \, u + \epsilon_2 \, v)^2 \Big) \, \tilde{\Psi}^{(i)} \; , \\ \tilde{\Psi}^{(i)*}_y &= -\tilde{\Psi}^{(i)*}_{xx} \; + 2 \Big(\, v + \epsilon_1 \, u + \epsilon_2 \, v \Big) \, \tilde{\Psi}^{(i)*}_x \\ &\qquad + \Big(\, 2 \, \epsilon_1 \, v_2 + \epsilon_2 \, (2 \, u + 2 \, v_x - v^2) - (\epsilon_1 \, u + \epsilon_2 \, v)^2 \Big) \, \tilde{\Psi}^{(i)*} \; . \end{aligned} \tag{6.12}$$

We now consider constraints between the fields v, u, v_2 etc. and square eigenfunctions, which turns (6.12) into a closed set of equations. The analog of the constraints (2.7) is given by [8]

$$(L^k)_{<1} = \sum_{i=1}^{m} \Psi^{(i)} \partial^{-1} \Psi^{(i)*} \partial = \sum_{i=1}^{m} \Psi^{(i)} \Psi^{(i)*} - \sum_{i=1}^{m} \Psi^{(i)} \partial^{-1} \Psi^{(i)*}_x \; , \tag{6.13}$$

where $\Psi^{(i)}$,$\Psi^{(i)*}$ solve (6.3). With the symmetry $m \leftrightarrow n$ in (6.8) one has

$$v_{t_k} = res(L \, \partial^{-1})_{t_k} = res(L^k \partial^{-1})_x \; , \quad u_{t_k} = res(L)_{t_k} = res(L^k)_x \; , \tag{6.14}$$

so that these constraints can be regarded as symmetry reductions

$$v_{t_k} = \Big(\sum_{i=1}^{m} \Psi^{(i)} \Psi^{(i)*} \Big)_x \; , \quad u_{t_k} = - \Big(\sum_{i=1}^{m} \Psi^{(i)} \Psi^{(i)*}_x \Big)_x \tag{6.15}$$

of the modified KP hierarchy and the KP hierarchy. The simplest case $k=1$ leads to

$$
\begin{aligned}
v &= \sum_{i=1}^{m} \Psi^{(i)}\Psi^{(i)*} &&= \sum_{i=1}^{m} \tilde{\Psi}^{(i)}\tilde{\Psi}^{(i)*} \ , \\
u &= -\sum_{i=1}^{m} \Psi^{(i)}\Psi^{(i)*}_x &&= -\sum_{i=1}^{m} \tilde{\Psi}^{(i)}\tilde{\Psi}^{(i)*}_x + \theta_x\, v \ , \\
v_2 &= \sum_{i=1}^{m} \Psi^{(i)}\Psi^{(i)*}_{xx} &&= \sum_{i=1}^{m} \tilde{\Psi}^{(i)}\tilde{\Psi}^{(i)*}_{xx} + 2\,\theta_x\, u - (\theta_x^2 + \theta_{xx})\, v \ .
\end{aligned} \tag{6.16}
$$

Insertion into (6.11.x) yields

$$
\theta_x = \epsilon_1\, \theta_x\, v - \epsilon_1 \sum_{i=1}^{m} \tilde{\Psi}^{(i)}\tilde{\Psi}^{(i)*}_x + \epsilon_2\, v \ , \tag{6.17}
$$

so that the gauge field is finally identified as

$$
\theta_x = \frac{1}{1-\epsilon_1\, v} \left(\epsilon_2\, v - \epsilon_1 \sum_{i=1}^{m} \tilde{\Psi}^{(i)}\tilde{\Psi}^{(i)*}_x \right) \tag{6.18}
$$

and the solution v of the modified KP equation as well as the solution u of the KP equation are expressed in terms of square eigenfunctions as

$$
v = \sum_{i=1}^{m} \tilde{\Psi}^{(i)}\tilde{\Psi}^{(i)*} \ , \quad u = \frac{1}{1-\epsilon_1\, v} \left(\epsilon_2\, v^2 - \sum_{i=1}^{m} \tilde{\Psi}^{(i)}\tilde{\Psi}^{(i)*}_x \right) \ . \tag{6.19}
$$

Together with a corresponding representation

$$
\begin{aligned}
v_2 \;=\; & \frac{\sum \tilde{\Psi}^{(i)}\tilde{\Psi}^{(i)*}_{xx}}{1-\epsilon_1\, v} + \epsilon_1\, \frac{v \sum \tilde{\Psi}^{(i)}_x\tilde{\Psi}^{(i)*}_x + 2\,(\sum \tilde{\Psi}^{(i)}\tilde{\Psi}^{(i)*}_x)^2}{(1-\epsilon_1\, v)^2} \\
& + \epsilon_2\, v\, \frac{\epsilon_2\, v^2 - v_x - 2 \sum \tilde{\Psi}^{(i)}\tilde{\Psi}^{(i)*}_x}{(1-\epsilon_1\, v)^2}
\end{aligned} \tag{6.20}
$$

equation (6.12) yields the first flow of an integrable hierarchy of nonlinear equations for the fields $\tilde{\Psi}^{(1)}, .., \tilde{\Psi}^{(m)*}$. These equations may be regarded as integrable two-parameter deformations of the vector Chen-Lee-Liu system [11]

$$
\begin{aligned}
\tilde{\Psi}^{(i)}_y &= \tilde{\Psi}^{(i)}_{xx} + 2 \Big(\sum_{j=1}^{m} \tilde{\Psi}^{(j)}\tilde{\Psi}^{(j)*} \Big)\, \tilde{\Psi}^{(i)}_x \ , \\
\tilde{\Psi}^{(i)*}_y &= -\tilde{\Psi}^{(i)*}_{xx} + 2 \Big(\sum_{j=1}^{m} \tilde{\Psi}^{(j)}\tilde{\Psi}^{(j)*} \Big) \tilde{\Psi}^{(i)*}_x \ ,
\end{aligned} \tag{6.21}
$$

obtained for $\epsilon_1 = \epsilon_2 = 0$. For $\epsilon_2 = 0$, $m=1$ and dropping the indices on the (adjoint) eigenfunctions one obtains the integrable equation

$$
\begin{aligned}
\tilde{\Psi}_y &= \tilde{\Psi}_{xx} + 2\,\tilde{\Psi}\tilde{\Psi}^*\tilde{\Psi}_x - \frac{2\,\epsilon_1\, \tilde{\Psi}^2\tilde{\Psi}^*_{xx}}{1-\epsilon_1\, \tilde{\Psi}\tilde{\Psi}^*} - \frac{2\,\epsilon_1\, \tilde{\Psi}\tilde{\Psi}_x\tilde{\Psi}^*_x + 3\,\epsilon_1^2\, \tilde{\Psi}^3\tilde{\Psi}^{*2}_x}{(1-\epsilon_1\, \tilde{\Psi}\tilde{\Psi}^*)^2} \ , \\
\tilde{\Psi}^*_y &= -\tilde{\Psi}^*_{xx} + 2\,\tilde{\Psi}\tilde{\Psi}^*\tilde{\Psi}^*_x - \frac{2\,\epsilon_1\, \tilde{\Psi}^*\tilde{\Psi}_x\tilde{\Psi}^*_x}{1-\epsilon_1\, \tilde{\Psi}\tilde{\Psi}^*} - \frac{2\,\epsilon_1\, \tilde{\Psi}\tilde{\Psi}^2_x - 3\,\epsilon_1^2\, \tilde{\Psi}^2\tilde{\Psi}^*\tilde{\Psi}^{*2}_x}{(1-\epsilon_1\, \tilde{\Psi}\tilde{\Psi}^*)^2} \ ,
\end{aligned} \tag{6.22}
$$

whose hierarchy provides solutions

$$v = \tilde{\Psi}\tilde{\Psi}^* \ , \quad u = -\frac{\tilde{\Psi}\tilde{\Psi}^*_x}{1 - \epsilon_1 \tilde{\Psi}\tilde{\Psi}^*} \tag{6.23}$$

of the (modified) KP hierarchy. This system seems to be a new member in the family of integrable equations with rational nonlinearities (see e.g. [16] and [17] for other examples). For $\epsilon_1 = 0$, $m = 1$ one obtains the integrable system

$$\begin{aligned} \tilde{\Psi}_y &= \tilde{\Psi}_{xx} + 2\,(1+\epsilon_2)\,\tilde{\Psi}\tilde{\Psi}^*\tilde{\Psi}_x + 2\,\epsilon_2\,\tilde{\Psi}^2\tilde{\Psi}^*_x + \epsilon_2(1-\epsilon_2)\,\tilde{\Psi}^3\tilde{\Psi}^{*2} \ , \\ \tilde{\Psi}^*_y &= -\tilde{\Psi}^*_{xx} + 2\,(1+\epsilon_2)\,\tilde{\Psi}\tilde{\Psi}^*\tilde{\Psi}^*_x + 2\,\epsilon_2\,\tilde{\Psi}^{*2}\tilde{\Psi}_x - \epsilon_2(1-\epsilon_2)\,\tilde{\Psi}^2\tilde{\Psi}^{*3} \ . \end{aligned} \tag{6.24}$$

leading to (modified) KP solutions via

$$v = \tilde{\Psi}\tilde{\Psi}^* \ , \quad u = \epsilon_2\, v^2 - \tilde{\Psi}\tilde{\Psi}^* \ . \tag{6.25}$$

For $\epsilon_2 = 1$ this flow reduces to the standard derivative NLS equation [9]

$$\begin{aligned} \tilde{\Psi}_y &= \tilde{\Psi}_{xx} + 2\,(\tilde{\Psi}^2\tilde{\Psi}^*)_x \ , \\ \tilde{\Psi}^*_y &= -\tilde{\Psi}^*_{xx} + 2\,(\tilde{\Psi}\tilde{\Psi}^{*2})_x \ , \end{aligned} \tag{6.26}$$

while for $\epsilon_2 = -1$ the Gerdjikov-Ivanov equation [10] is obtained:

$$\begin{aligned} \tilde{\Psi}_y &= \tilde{\Psi}_{xx} + 2\,\tilde{\Psi}^2\tilde{\Psi}^*_x - 2\,\tilde{\Psi}^3\tilde{\Psi}^{*2} \ , \\ \tilde{\Psi}^*_y &= -\tilde{\Psi}^*_{xx} + 2\,\tilde{\Psi}^{*2}\tilde{\Psi}_x + 2\,\tilde{\Psi}^2\tilde{\Psi}^{*3} \ . \end{aligned} \tag{6.27}$$

Thus the constrained modified KP hierarchy is equivalent to the hierarchy of generalized derivative NLS equations (6.12)/(6.19)/(6.20) which reduces to different other known hierarchies depending on the gauge choice.

VII Concluding remarks

The gauge freedom in the constraints of the KP and the modified KP hierarchy can be exploited effectively to generate integrable deformations of 1+1-dimensional systems. It was demonstrated that a variety of well-established integrable hierarchies can be connected to the constrained KP flows via gauge transformations.

Two qualitatively different kinds of deformations are obtained. In the isotropic vector generalization the global symmetry of the original system is preserved, while for anisotropic multicomponent extensions this symmetry is broken. The preservation of integrability despite of the symmetry breaking is a significant fact, since commonly one encounters a different picture [18].

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