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Matrix Sato Theory and Integrable Equations in 2+1 Dimensions

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ABSTRACT

Two different decompositions of the algebra of matrix valued pseudo-differential symbols are considered. The two associated classes of nonlinear integrable equations in 2+1 dimensions are discussed within the framework of generalized Lax equations and Sato's approach. The two classes represent the multi-component KP hierarchy and a "modified" multi-component KP hierarchy. It is shown that general Darboux theorems provide Miura transformations between these two classes as well as auto-Bäcklund transformations within the classes.

1 Introduction

Sato made a remarkable contribution to our understanding of the structure of soliton equations. He developed the theory of the Kadomtsev-Petviashvili (KP) hierarchy and revealed its algebraic structure[1−5]. The KP equation itself resides in this framework as well as many other soliton systems such as the KdV or the Boussinesq equation[3−5]. In its simplest form the KP hierarchy describes integrable equations associated with scattering problems given by scalar differential operators. Sato also introduced a multi-component version of this framework[1,2], in which 2+1 dimensional integrable equations such as the Davey-Stewartson (DS) can be identified[3,6]. In this algebraic context gauge transformations for the underlying pseudo-differential Lax operators were recently discussed for the scalar case[7,8].

In this note, we consider the multi-component KP hierarchy. We show how Miura- and auto-Bäcklund transformations reside in Sato's construction via gauge transformations of the underlying Lax operators. In contrast to the scalar case[7,8], where Lax operators are suitable tools, it is more convenient to use dressing operators in the multi-component version.

In Section 2 we briefly review the construction of the multi-component KP hierarchy and a modified multi-component hierarchy. It provides a framework to derive various integrable 2+1 dimensional equations with their hierarchies of commuting symmetries and non-commuting (time-dependent) master symmetries[9]. In Section 3 we specify the construction and identify a number of soliton hierarchies related to the KP, modified KP, DS, Ishimori, N-wave and "modified N-wave" equations. Section 4 is devoted to transformations of the dressing operators leading to Miura- and auto-Bäcklund transformations for these hierarchies.

2 Background

We start with a collection of "undressed operators" $\mathcal{A} = \{A\}$, where we will mainly be interested in operators of the form $A = A_n = a_n \partial^n$, $n \in \mathbb{N}$. Here the coefficients a_n are assumed to take values in some finite dimensional (matrix) algebra g_0. We consider two classes of dressing operators –labeled as "$k = 0$" and "$k = 1$"– defined as formal expansions of the form

$$\begin{aligned} \underline{k=0}: &\qquad W = 1 + w_1 \partial^{-1} + w_2 \partial^{-2} + .. \ , \\ \underline{k=1}: &\qquad W = w_0 + w_1 \partial^{-1} + w_2 \partial^{-2} + .. \ . \end{aligned} \tag{2.1}$$

Here the coefficients $(w_1, w_2, ..)$ and $(w_0, w_1, ..)$, respectively, are functions of a variable x taking values in the same (matrix) algebra g_0 used for the undressed operators. The pseudo-differential symbol in (2.1) is defined via the algebraic rule $\partial^{-1} f = f\partial^{-1} - f_x \partial^{-2} + f_{xx}\partial^{-3} - ..$ generalizing the Leibniz rule $\partial^n f = \sum_{i \geq 0} \binom{n}{i} \frac{\partial^i f}{\partial x^i} \partial^{n-i}$ for differential operators. Assuming w_0 in (2.1.ii) to be invertible we can formally invert these operators by an expansion

$$\begin{aligned} \underline{k=0}: \ W^{-1} &= 1 - w_1 \partial^{-1} + (w_1^2 - w_2)\partial^{-2} + .., \\ \underline{k=1}: \ W^{-1} &= w_0^{-1} - w_0^{-1} w_1 w_0^{-1} \partial^{-1} + .. \ . \end{aligned} \tag{2.2}$$

To each undressed operator $A \in \mathcal{A}$ we associate a time variable t_A and a dynamics of the dressing operators via

$$\frac{dW}{dt_A} = W (W^{-1} A W)_{<k} \ , \quad k = 0 \text{ or } 1 \ , \tag{2.3}$$

where $(\sum_i u_i \partial^i)_{<k} = \sum_{i<k} u_i \partial^i$ is the projection to differential orders smaller than k. We note that the form (2.1) of the dressing operators is left invariant by the dynamics (2.3) for both cases $k = 0$ and $k = 1$. For the special choice $A = 1\partial$ one finds $dW/dt_{1\partial} = W_x$, so that the variable x can be identified with the time $t_{1\partial}$. Defining $M(A) = (W^{-1} A W)_{\geq k}$ we consider the compatibility of two flows generated by two undressed operators A and $\tilde{A}$ and find

$$\frac{dM(A)}{dt_{\tilde{A}}} - \frac{dM(\tilde{A})}{dt_A} - [M(\tilde{A}), M(A)] = M\left(\frac{dA}{dt_{\tilde{A}}} - \frac{d\tilde{A}}{dt_A} - [\tilde{A}, A]\right) \ . \tag{2.4}$$

Here we have included the case that the undressed operators themselves may depend on the time variables, too. The simple proof of this identity makes only use of the fact that in both cases $k = 0$ and $k = 1$ the subspaces $g_{\geq k} = \{\sum_{i \geq k} u_i \partial^i\}$ and $g_{<k} = \{\sum_{i<k} u_i \partial^i\}$ form proper (Lie) subalgebras in the space of all pseudo-differential symbols. As a result, the compatibility of the two flows (2.3) generated by A and $\tilde{A}$ is given by

$$\frac{d}{dt_{\tilde{A}}}\frac{dW}{dt_A} - \frac{d}{dt_A}\frac{dW}{dt_{\tilde{A}}} = W \left(W^{-1} \left(\frac{dA}{dt_{\tilde{A}}} - \frac{d\tilde{A}}{dt_A} - [\tilde{A}, A]\right) W \right)_{<k} \quad . \tag{2.5}$$

We conclude that the hierarchy of evolution equations (2.3) generated by a collection of undressed operators commutes, if all pairs $A, \tilde{A}$ in this collection satisfy the zero-curvature conditions $\frac{dA}{dt_{\tilde{A}}} - \frac{d\tilde{A}}{dt_A} = [\tilde{A}, A]$. According to (2.4) this condition then is inherited to $M(A)$ and $M(\tilde{A})$. In particular, if we choose undressed operators of the form $\mathcal{A} = \{ \ a_n \partial^n \ , \ n \in \mathbb{N} \ \}$ with commuting constant (=time independent) matrices a_n, then we obtain a hierarchy of commuting equations

$$\frac{dW}{dt_n} = W(W^{-1} a_n \partial^n W)_{<k} \quad , \quad k = 0 \text{ or } 1 \quad , \tag{2.6}$$

for the infinite set of fields $(w_1, w_2, ..)$ and $(w_0, w_1, ..)$ parametrizing the dressing operators (2.1). The zero-curvature equations (2.4) then guarantee the compatibility of the associated linear scattering problems

$$\Phi_{t_n} = M(a_n \partial^n)\Phi = (W^{-1} a_n \partial^n W)_{\geq k} \Phi \quad . \tag{2.7}$$

For the hierarchy (2.6) of non-linear equations one can also construct a hierarchy of time-dependent symmetries. For instance, considering the undressed operators

$$M_m^{(1)} = \sum_{l \geq 1} l\, t_l\, a_l\, \partial^{l+m} \quad , \quad M_m^{(2)} = \sum_{l_1, l_2 \geq 1} l_1 l_2\, t_{l_1} t_{l_2}\, a_{l_1} a_{l_2}\, \partial^{l_1 + l_1 + m} + \sum_{l \geq 2} l(l-1)\, t_l\, a_l\, \partial^{l+m} \quad , \tag{2.8}$$

is is easily checked that the zero-curvature equations $\frac{d}{dt_n} M_m^{(d)} = [a_n \partial^n, M_m^{(d)}]$, $d = 1, 2$, with the constant commuting $a_n \partial^n$ are satisfied. Here one has assumed $a_1 = 1$ and it is crucial that the corresponding time t_1 is identified with the variable x defining the differential symbols. Hence, in addition to the hierarchy (2.6) we can consider flows $\frac{dW}{dt_{md}} = W(W^{-1} M_m^{(d)} W)_{<k}$, where t_{md} denotes the time associated with $M_m^{(d)}$. They will commute with all flows given by (2.6) (but not among themselves). It is well known that the KP is a special realization of (2.6). The vector fields generated by the time dependent undressed operators $M_m^{(d)}$ correspond to the master symmetries discussed in Ref. 9. In particular, $M_m^{(1)}$ and $M_m^{(2)}$ generate master symmetries of degree 1 and 2, respectively. Compact formulas such as (2.8) for the generating time polynomials can be obtained for master symmetries of any degree d.

We remark that the Sato equation (2.3) immediately leads to the Lax equations

$$\frac{dL_B}{dt_A} = [(L_A)_{\geq k}, L_B] + W^{-1}(\frac{dB}{dt_A} - [A, B])W \quad , \tag{2.9}$$

when we consider Lax operators $L_A = W^{-1}AW$, $L_B = W^{-1}BW$, obtained by dressing operators $A, B \in \mathcal{A}$. Usually[3–5], for $A = A_n = a_n\partial^n$ with constant a_n the simplest choice $B = \partial$ is used to define the Lax operator $L = L_B$ satisfying the hierarchy of Lax equations $\frac{d}{dt_A}L = [(L_A)_{\geq k}, L]$. For the flows generated by the operators $M_m^{(d)}$ above, a Lax formulation is given by fixing $L = W^{-1}\partial W$, say, and choosing $A = M_m^{(d)}$. Hence, the master symmetries of the KP hierarchy (and other hierarchies) admit the Lax representations $\frac{d}{dt_{md}}L = [(W^{-1}M_m^{(d)}W)_{\geq k}, L] + W^{-1}(M_m^{(d)})_x W$.

3 Realizations

We consider undressed operators $\mathcal{A} = \{A_n = a_n\partial^n \ , \ n \in N\}$, where the (time-independent) coefficients a_n are selected from a commuting subset of all the $N \times N$-matrices. In particular, the a_n will be chosen as $N \times N$ diagonal matrices.

$\underline{N = 1, \ k = 0 \ :}$

Choosing $A_2 = \partial^2$, $A_3 = \partial^3$ and denoting the corresponding times by $t_2 = y, \ t_3 = t$, the dynamics (2.6) for the operator (2.1.i) translates to

$$\begin{aligned} \frac{d}{dy}\begin{pmatrix} w_1 \\ w_2 \\ .. \end{pmatrix} &= \begin{pmatrix} w_{1xx} + 2w_{2x} - 2w_1w_{1x} \\ w_{2xx} + 2w_{3x} + 2w_1w_{1xx} - 2w_{1x}w_2 \\ .. \end{pmatrix} \quad , \\ \frac{d}{dt}\begin{pmatrix} w_1 \\ .. \end{pmatrix} &= \begin{pmatrix} w_{1xxx} + 3w_{2xx} + 3w_{3x} + 3w_1^2w_{1x} - 3(w_1w_2)_x \\ .. \end{pmatrix} \quad . \end{aligned} \tag{3.1}$$

Using (3.1.y) one eliminates w_2 and w_3 in (3.1.t) and obtains the potential KP equation[1–5]

$$4w_{1tx} = w_{1xxxx} + 12w_{1x}w_{1xx} + 3w_{1yy} \quad . \tag{3.2}$$

Alternatively, one may consider the scattering operators $M(A) = (W^{-1}AW)_{\geq 0}$ calculated as

$$M(\partial^2) = \partial^2 + 2w_{1x} \quad , \quad M(\partial^3) = \partial^3 + 3w_{1x}\partial + 3(w_{1xx} + w_{2x} - w_1w_{1x}) \tag{3.3}$$

and derive (3.2) directly from the zero-curvature condition (2.4).

$\underline{N = 1, \ k = 1 \ :}$

We choose $A_2 = \partial^2$, $A_3 = \partial^3$ and again denote the corresponding times by $t_2 = y$, $t_3 = t$. The scattering operators

$$M(\partial^2) = \partial^2 + 2{w_0}^{-1} w_{0x}\partial \quad , \quad M(\partial^3) = \partial^3 + 3{w_0}^{-1} w_{0x}\partial^2 + 3({w_0}^{-1} w_{0xx} + (w_0^{-1} w_1)_x)\partial \tag{3.4}$$

suggest to introduce the field $v = {w_0}^{-1} w_{0x}$. In terms of v the zero-curvature condition (2.4) for (3.4) leads to the modified KP equation

$$4v_{tx} = (v_{xxx} - 6v^2 v_x)_x + 3v_{yy} + 6v_x v_y + 6v_{xx} D^{-1} v_y \quad , \tag{3.5}$$

where $D^{-1} = \int^x$ is an integration originating from the elimination of the auxiliary variable w_1.

<u>$N = 2,\ k = 0$:</u>

We choose $A_1 = \sigma_3 \partial$, $A_2 = \sigma_3 \partial^2$ and denote the corresponding times by y and t, respectively. Here $\sigma_3 = diag(1, -1)$ is the third Pauli matrix. We directly proceed to the scattering operators

$$M(\sigma_3\partial) = \sigma_3\partial + P \quad , \quad M(\sigma_3\partial^2) = \sigma_3\partial^2 + P\partial + Q \quad , \tag{3.6}$$

where $P = [\sigma_3, w_1]$ and $Q = w_{1y} + \sigma_3 w_{1x}$. In this identification of Q we have already eliminated the field w_2 using the y-evolution (2.3). The zero-curvature conditions (2.4) leads to

$$P_y + \sigma_3 P_x = [\sigma_3, Q] \quad , \quad P_t - Q_y = \sigma_3 P_{xx} - \sigma_3 Q_x + P P_x + [Q, P] \quad . \tag{3.7}$$

Splitting $Q = Q^{diag} + Q^{off}$ in a diagonal and an off-diagonal part, one can eliminate Q^{off} from (3.7) and derives the Davey-Stewartson equation

$$2P_t = \sigma_3(P_{xx} + P_{yy}) + 2\sigma_3 P\, tr(\sigma_3 Q^{diag}) \quad , \quad 2(\sigma_3 Q_x^{diag} - Q_y^{diag}) = \sigma_3 (P^2)_y + (P^2)_x \tag{3.8}$$

for the off-diagonal matrix field $P = [\sigma_3, w_1]$.

<u>$N = 2,\ k = 1$:</u>

We again choose $A_1 = \sigma_3 \partial$, $A_2 = \sigma_3 \partial^2$ and denote the corresponding times by y and t, respectively. We directly proceed to the scattering operators

$$M(\sigma_3\partial) = S\partial \quad , \quad M(\sigma_3\partial^2) = S\partial^2 + T\partial \quad , \tag{3.9}$$

where $S = {w_0}^{-1}\sigma_3 w_0$ and $T = w_0^{-1}\sigma_3 w_{0x} + w_0^{-1} w_{0y}$. In this identification of T we have already eliminated the field w_1 using the y-evolution

$$w_{0y} = \sigma_3 w_{0x} + w_0[S, w_0^{-1} w_1] \tag{3.10}$$

obtained from (2.3). The zero-curvature conditions (2.4) leads to

$$S_y + S S_x = [S,T] \quad , \quad S_t - T_y = S S_{xx} + T S_x - S T_x \quad . \tag{3.11}$$

Using the three Pauli matrices $\sigma_{1,2,3}$ we write $S = S_1\sigma_1 + S_2\sigma_2 + S_3\sigma_3$ (observing that $S = w_0^{-1}\sigma_3 w_0$ is traceless) and $T = T_1\sigma_1 + T_2\sigma_2 + T_3\sigma_3 + \beta\,1$. We form vectors $\vec{S}$ and $\vec{T}$ from the components S_i and T_i, respectively. Using (3.10) we observe $tr(w_0^{-1}w_{0y}) = tr(w_0^{-1}\sigma_3 w_{0x})$ and $tr(w_0^{-1}w_{0x}) = tr(w_0^{-1}\sigma_3 w_{0y})$, so that

$$\beta = \frac{1}{2}tr(T) = tr(w_0^{-1}w_{0y}) = \phi_y \ , \ \vec{S}\cdot\vec{T} = \frac{1}{2}tr(ST) = tr(w_0^{-1}w_{0x}) = \phi_x \ , \tag{3.12}$$

where $\phi = ln(det\, w_0)$. Observing $S^2 = 1$ one finds $\vec{S}\cdot\vec{S} = \frac{1}{2}tr(S^2) = 1$ and (3.11) can be transformed into the Ishimori equation

$$\vec{S}_t = \frac{i}{2}\vec{S}\times(\vec{S}_{xx}+\vec{S}_{yy}) + \phi_y\vec{S}_x + \phi_x\vec{S}_y \quad , \quad \phi_{yy} - \phi_{xx} = i\,\vec{S}\cdot(\vec{S}_x\times\vec{S}_y) \quad . \tag{3.13}$$

$\underline{N \geq 3,\ k = 0\ :}$

We choose two diagonal $N\times N$-matrices $a = diag(a_1,..,a_N)$ and $\tilde{a} = diag(\tilde{a}_1,..,\tilde{a}_N)$. To $A = a\partial$ and $\tilde{A} = \tilde{a}\partial$ we associate the times $t_A = y$ and $t_{\tilde{A}} = t$. The scattering operators are calculated as $M(A) = a\partial + P$, $M(\tilde{A}) = \tilde{a}\partial + \tilde{P}$ with $P = [a, w_1]$ and $\tilde{P} = [\tilde{a}, w_1]$, respectively. The zero-curvature conditions (2.4) leads to

$$[a,\tilde{P}] = [\tilde{a}, P] \quad , \quad P_t = \tilde{P}_y + \tilde{a}P_x - a\tilde{P}_x + [\tilde{P}, P] \quad . \tag{3.14}$$

Observing that $\overset{(\sim)}{P} = [\overset{(\sim)}{a}, w_1]$ are off-diagonal, the first constraint leads to $\tilde{P}_{ij} = \frac{\tilde{a}_i-\tilde{a}_j}{a_i-a_j}P_{ij}$, provided that the a_i's are distinct. The t-evolution (3.14) then produces the 2+1 dimensional N-wave equation. Introducing further flows generated by operators of the form $a\partial$ with diagonal a's leads to a set of relations like (3.14), which can be rewritten as a multi-dimensional N-wave equation.

$\underline{N \geq 3,\ k = 1\ :}$

We again choose two diagonal $N\times N$-matrices $a = diag(a_1,..,a_N)$ and $\tilde{a} = diag(\tilde{a}_1,..,\tilde{a}_N)$ and associate the times $t_A = y$ and $t_{\tilde{A}} = t$ to the operators $A = a\partial$ and $\tilde{A} = \tilde{a}\partial$, respectively. The scattering operators now are calculated as $M(A) = S\partial$, $M(\tilde{A}) = \tilde{S}\partial$ with $S = w_0^{-1}aw_0$ and $\tilde{S} = w_0^{-1}\tilde{a}w_0$, respectively. The zero-curvature conditions (2.4) leads to

$$[S,\tilde{S}] = 0 \quad , \quad S_t - \tilde{S}_y = \tilde{S}S_x - S\tilde{S}_x \quad . \tag{3.15}$$

For distinct eigenvalues $a_1,..,a_N$ of S we can express $\tilde{a} = \sum_{n=0}^{N-1} c_n a^n$ with some constants c_n, so that with $\tilde{S} = \sum_{n=0}^{N-1} c_n S^n$ we obtain the "modified" N-wave equation

$$S_t = \sum_{n=0}^{N-1} c_n \left((S^n)_y + S^n S_x - S(S^n)_x \right) \quad . \tag{3.16}$$

A discussion of this local integrable 2+1 dimensional model and some of its reductions can be found in Ref. 10.

4 Darboux and Bäcklund transformations

Invariances of and relations among the various non-linear equations encoded in the Sato equations (2.3)/(2.6) are built into their construction via the pseudo-differential dressing operator W. The key to these links are "eigenfunctions" Φ satisfying the linear scattering problems (2.7). With $t = t_A$ we again consider the evolution equation $\frac{d}{dt}W = W(W^{-1}AW)_{<k}$ generated by an undressed operator A. A (matrix-valued) function Φ satisfying $\Phi_t = (W^{-1}AW)_{\geq k}\Phi$ shall be called an eigenfunction for this equation. It was shown in Refs. 7,8, how such eigenfunctions can be used in gauge transformations $L \to \hat{L} = T(\Phi)LT^{-1}(\Phi)$ of the associated Lax operators to build up a very general picture on Miura- and auto-Bäcklund transformations inside (scalar) Sato theory. Here $T(\Phi)$ is in general a (pseudo-)differential gauge operator depending on the eigenfunction. Considering the multi-component version, we have used the more convenient definition of the dynamics in terms of dressing operators rather than in terms of Lax operators. Using $L = W^{-1}\partial W$, say, the scalar results can be translated to transformations $W \to \hat{W} = WT^{-1}(\Phi)$, so that $L = W^{-1}\partial W \to \hat{L} = \hat{W}^{-1}\partial\hat{W}$ reproduces the gauge transformations of Refs. 7,8.

Theorem 1 (Darboux transformations):
Let B be any time-independent operator commuting with A.

a) If $\frac{d}{dt}W = W(W^{-1}AW)_{<0}$ and $\Phi_t = (W^{-1}AW)_{\geq 0}\Phi$, then $\hat{W} = BW\Phi\partial^{-1}\Phi^{-1}$ satisfies $\frac{d}{dt}\hat{W} = \hat{W}(\hat{W}^{-1}A\hat{W})_{<0}$,

b) if $\frac{d}{dt}W = W(W^{-1}AW)_{<0}$ and $\Phi_t = (W^{-1}AW)_{\geq 0}\Phi$, then $\hat{W} = BW\Phi$ satisfies $\frac{d}{dt}\hat{W} = \hat{W}(\hat{W}^{-1}A\hat{W})_{<1}$,

c) if $\frac{d}{dt}W = W(W^{-1}AW)_{<1}$ and $\Phi_t = (W^{-1}AW)_{\geq 1}\Phi$, then $\hat{W} = BW\Phi$ satisfies $\frac{d}{dt}\hat{W} = \hat{W}(\hat{W}^{-1}A\hat{W})_{<1}$,

d) if $\frac{d}{dt}W = W(W^{-1}AW)_{<1}$ and $\Phi_t = (W^{-1}AW)_{\geq 1}\Phi$, then $\hat{W} = BW\partial^{-1}\Phi_x$ satisfies $\frac{d}{dt}\hat{W} = \hat{W}(\hat{W}^{-1}A\hat{W})_{<1}$,

e) if $\frac{d}{dt}W = W(W^{-1}AW)_{<1}$ and $\Phi_t = (W^{-1}AW)_{\geq 1}\Phi$, then $\hat{W} = BW\Phi\partial^{-1}(\Phi^{-1})_x$ satisfies $\frac{d}{dt}\hat{W} = \hat{W}(\hat{W}^{-1}A\hat{W})_{<1}$.

An important feature of these transformations is that they provide a natural mechanism to map further eigenfunctions Ψ of the equations under consideration to new eigenfunctions $\hat{\Psi} = T(\Phi)\Psi$ of the transformed equations. This leads to surprisingly simple relations among the Bianchi diagrams of these transformations[7,8].

For instance, considering case a), one has the result that a further eigenfunction Ψ satisfying $\Psi_t = (W^{-1}AW)_{\geq 0}\Psi$ is mapped to a new eigenfunction $\hat{\Psi} = \Phi(\Phi^{-1}\Psi)_x$ for $\hat{W}$ satisfying $\hat{\Psi}_t = (\hat{W}^{-1}A\hat{W})_{\geq 0}\hat{\Psi}$. Similar statements hold for all other cases, too.

We may choose the arbitrary operators B in such a way that the form (2.1) of the dressing operators is kept invariant under the transformation. As shown before, the highest coefficients w_1 and w_0 satisfy a hierarchy of integrable non-linear equations in 2+1 dimensions. Hence, calculating the action of these transformations on these highest coefficients yields transformations of these fields in terms of eigenfunctions, which can be eliminated using the eigenfunction equation. Thus case a) describes an invariance of the multi-component KP hierarchy, case b) provides the Miura-link between the multi-component KP hierarchy and the modified multi-component KP hierarchy, cases c)-e) yields various invariances of the modified multi-component KP hierarchy. It can be shown[7,8] that the transformations c) and d) are "elementary" transformations, which can be composed to the final transformation e).

The scalar cases related to the KP and the modified KP equations were treated in Ref. 7. Here, we demonstrate the procedure for the DS equation (3.8). Using Theorem 1.a) with $B = \partial$ one calculates $\hat{W} = \partial W \Phi \partial^{-1}\Phi^{-1} = 1 + (w_1 + \Phi_x\Phi^{-1})\partial^{-1} + \ldots$. Here, the special choice for B is indicated, to keep the form $\hat{W} = 1 + ..$ of the dressing operator. According to Theorem 1 the evolution (2.3) for $\hat{W} = 1 + \hat{w}_1\partial^{-1} + \ldots$ with $\hat{w}_1 = w_1 + \Phi_x\Phi^{-1}$ encodes the DS hierarchy, if we assume that W satisfies the Sato equation (2.6) with a corresponding undressed operator. With $P = [\sigma_3, w_1]$, $Q = w_{1y} + \sigma_3 w_{1x}$, $\hat{P} = [\sigma_3, \hat{w}_1]$, and $\hat{Q} = \hat{w}_{1y} + \sigma_3\hat{w}_{1x}$, the two DS solutions (P, Q) and $(\hat{P}, \hat{Q})$ are related by

$$\hat{P} = P + [\sigma_3, \Phi_x\Phi^{-1}] \ , \ \hat{Q} = Q + (\Phi_x\Phi^{-1})_y + \sigma_3(\Phi_x\Phi^{-1})_x \ , \ \Phi_y = \sigma_3\Phi_x + P\Phi \ . \tag{4.1}$$

The last relation is the required eigenfunction condition $\Phi_y = (W^{-1}\sigma_3\partial W)_{\geq 0}\Phi$ given by the operator $M(\sigma_3\partial)$ in (3.6). Elimination of the eigenfunction Φ yields the "spatial" part of an auto-Bäcklund transformation for the DS hierarchy. We decompose $\Phi_x\Phi^{-1}$ and $\Phi_y\Phi^{-1}$ in a diagonal and an off-diagonal part. The diagonal and off-diagonal part of the equations (4.1) yield

$$\begin{array}{ll} (\Phi_x\Phi^{-1})^{off} = \frac{1}{2}\sigma_3(\hat{P} - P) & , \ (\Phi_x\Phi^{-1})^{diag}_y + \sigma_3(\Phi_x\Phi^{-1})^{diag}_x = \hat{Q}^{diag} - Q^{diag} \ , \\ (\Phi_y\Phi^{-1})^{diag} - \sigma_3(\Phi_x\Phi^{-1})^{diag} = 0 & , \ (\Phi_y\Phi^{-1})^{off} = \frac{1}{2}(\hat{P} + P) \ . \end{array} \tag{4.2}$$

With

$$((\Phi_y\Phi^{-1})_x - (\Phi_x\Phi^{-1})_y)^{off} = [(\Phi_x\Phi^{-1})^{off}, (\Phi_y\Phi^{-1})^{diag}] + [(\Phi_x\Phi^{-1})^{diag}, (\Phi_y\Phi^{-1})^{off}] \tag{4.3}$$

one obtains the auto-Bäcklund transformation

$$(\hat{P} + P)_x - \sigma_3(\hat{P} - P)_y = 2DP - 2\hat{P}D \ , \ \hat{Q}^{diag} - Q^{diag} = D_y + \sigma_3 D_x \ , \tag{4.4}$$

with the auxiliary diagonal matrix $D = (\Phi_x \Phi^{-1})^{diag}$.

In a similar fashion all statements of Theorem 1 can be used to characterize (auto-) Bäcklund transformations of the non-linear equations encoded in the Sato equations (2.3). In particular, case b) describes the Miura transformations between the equations of the class $k = 0$ to corresponding equations in the class $k = 1$. We remark that with $\hat{W} = W\Phi = (1 + w_1\partial^{-1} + \ldots)\Phi = \Phi + w_1\Phi\partial^{-1} + \ldots$ the leading coefficient w_0 in (2.1) for the class $k = 1$ can always be regarded as the eigenfunction of the corresponding equation in the class $k = 0$. For instance, as we have identified the DS equation (3.8) and the Ishimori equation (3.13) as representatives of the two different classes related to the same undressed operators, we obtain a Miura transformation in the form

$$S = \Phi^{-1}\sigma_3\Phi \quad , \quad \phi = ln(\det \Phi) \quad , \quad \Phi_y = \sigma_3\Phi_x + P\Phi \quad , \tag{4.5}$$

from which Φ has to be eliminated.

In a similar way all transformations of Theorem 1 are generated by an eigenfunction, which has to be eliminated to obtain the Bäcklund relations. We remark that this elimination process can be described on a general level. For instance, considering case a) we have $\hat{W} = BW\Phi\partial^{-1}\Phi^{-1}$, which can be rewritten as $\hat{W}^{-1}BW = \Phi\partial\Phi^{-1} =$ *purely differential*. In fact, a direct link between the coefficients of $\hat{W}$ and W is obtained by imposing the constraint $(\hat{W}^{-1}BW)_{<0} = 0$. Although motivated by the transformations of Theorem 1, this result can immediately be generalized to

Theorem 2 (Bäcklund transformations):

Let

$$\frac{d}{dt}W = W(W^{-1}AW)_{<k} \quad , \quad \frac{d}{dt}\hat{W} = \hat{W}(\hat{W}^{-1}A\hat{W})_{<\hat{k}} \quad , \quad k, \hat{k} \in \{0, 1\}. \tag{4.6}$$

If $\hat{k} \geq \tilde{k} \geq 0$, then for arbitrary constant operator B commuting with A the constraint $(\hat{W}^{-1}BW)_{<\tilde{k}} = 0$ is invariant under the dynamics (4.6).

Choosing first order differential operators B and $k = \hat{k}$ this simple result gives a unified description of the Bäcklund transformations for the various realizations of the Sato equation (2.3) as given in Section 3.

We remark that one can impose the additional constraint $\hat{W} = W$, $\hat{k} = k$, so that $(W^{-1}BW)_{<k} = 0$ yields relations between the infinitely many fields parametrizing the dressing operator W. Now, choosing an n-th order differential operator B, only $n - 1$ of the fields in W will be independent. All "higher" fields can be expressed as differential polynomials of these first $n - 1$ fields. The Sato equations (2.3) then lead to a closed system of non-linear equations in $1 + 1$ dimensions for these fields. According to Theorem 2, this process ("n-reduction") of reducing the Sato-equations

originates from the same background as the Miura- and auto-Bäcklund transformations.

Another way of obtaining 1+1 dimensional equations is to assume that all coefficients $(w_1, \ldots)$, $(w_0, \ldots)$ of the dressing operators (2.1) are independent of the space variable x defining the symbol ∂. We consider the associated Lax operators $L_B = W^{-1}BW = \sum u_m \partial^m$ and perform Fourier transformation to $Q(L_B) = \frac{1}{2\pi}\int\int e^{i\lambda x} L e^{-i\mu x} d\mu dx = \frac{1}{2\pi}\int\int \sum_m u_m e^{i(\lambda-\mu)x}(-i\mu)^m d\mu dx = \sum_m u_m(-i\lambda)^m$. The projection $(\ldots)_{\geq k}$ to differential orders greater or equal to k now become projections $(\ldots)_{\geq k}$ to powers of λ greater or equal to k, i. e. $Q((L_B)_{\geq k}) = (Q(L_B))_{\geq k}$. The Lax equation $\frac{d}{dt_A} L_B = [(L_A)_{\geq k}, L_B]$ generated by an undressed operator A translate to new Lax equations $\frac{d}{dt_A} Q(L_B) = [(Q(L_A)_{\geq k}, Q(L_B)]$ on the loop space $g_0 \otimes C(\lambda, \lambda^{-1})$ of the matrix algebra g_0, from which the coefficients of the dressing operators (2.1) were taken. With appropriate choices of A and B we thus have made the connection with the results of Ref. 11, where a unified description of the integrable 1+1 dimensional equations connected with $sl(2)$ was given.

References

[1] M. Sato, *RIMS Kôkyuroku, Kyoto Univ.* **439** (1981), 30.

[2] M. Sato, Y. Sato, in *Nonlinear Partial Differential Equations in Applied Science*, Lecture Notes in Numerical and Applied Analysis **5**, eds. H. Fujita, P.D. Lax, G. Strang (Kinokuniya/North Holland, Tokyo, 1982), 259.

[3] M. Jimbo, T. Miwa, *Publ. RIMS, Kyoto Univ.* **19** (1983), 943.

[4] E. Date, M. Jimbo, K. Kashiwara, T. Miwa, in *Nonlinear Integrable Systems - Classical Theory and Quantum Theory*, eds. M. Jimbo, T. Miwa, (World Scientific Publ, Singapore, 1983), 39.

[5] Y. Ohta, J. Satsuma, D. Takahashi, T. Tokihiro, *Progr. Theor. Phys. Suppl.* **94** (1988), 210.

[6] K. Kajiwara, J. Matsukidaira, J. Satsuma, *Phys. Lett.* **146A** (1990), 115.

[7] W. Oevel, C. Rogers, *Gauge Transformations and Reciprocal Links in 2+1 Dimensions*, preprint CDSNS91-51, Georgia Institute of Technology (1991).

[8] B. G. Konopelchenko, W. Oevel, *An r-Matrix Approach to Nonstandard Classes of Integrable Equations*, preprint Mathematical Sciences Report No. A 149, Loughborough University (1991).

[9] B. Fuchssteiner, *Progr. Theor. Phys.* **68** (1982), 1082.

[10] B. G. Konopelchenko, *Reviews in Math. Phys.* **2** (1990), 399.

[11] H. Flaschka, A. C. Newell, T. Ratiu, *Physica* **9D** (1983), 300.