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Gauge Transformations and Reciprocal Links in 2+1 Dimensions

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Abstract

Generalized Lax equations are considered in the spirit of Sato theory. Three decompositions of an underlying algebra of pseudo-differential operators lead, in turn, to three different classes of integrable nonlinear hierarchies. These are associated with Kadomtsev-Petviashvili, modified Kadomtsev-Petviashvili and Dym hierarchies in 2+1 dimensions. Miura- and auto-Bäcklund transformations are shown to originate naturally from gauge transformations of the Lax operators. General statements on reciprocal links between these hierarchies are established, which, in particular, give rise to novel reciprocal auto-Bäcklund transformations for the Dym hierarchy. These links are formulated as Darboux theorems for the associated Lax operators.

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1 Introduction

The inverse scattering transform (IST) is by now a well-established method for the treatment of initial value problems for a broad class of nonlinear evolution equations [1-6]. A key feature of the IST method is the correspondence between a **nonlinear** system and an associated pair of **linear** problems

$$L_1\Psi = 0 \quad , \quad L_2\Psi = 0 \quad , \tag{1.1}$$

where the operators L_1 and L_2 are parametrized by a certain set of functions $u_1(x_1,..,x_m)$,.., $u_n(x_1,..,x_m)$ of the independent variables $x_1,..,x_m$. Compatibility of the two linear problems (1.1), namely the commutativity condition

$$[L_1, L_2] = 0 \quad , \tag{1.2}$$

produces the nonlinear systems for the u_i which are amenable to the IST. The standard IST method involves the application of both direct and inverse spectral methods to the linear systems (1.1) to solve initial value problems for the associated **integrable** nonlinear system.

Gauge (dressing) transformations

$$\Psi \ \rightarrow \ \Psi' = G\Psi \quad , \quad L_i \ \rightarrow \ L'_i = GL_iG^{-1} = L'_i(u'_1,..,u'_n) \ , \ i = 1,2 \ , \tag{1.3}$$

applied to the inverse scattering formalism (1.1) were originally introduced by Zakharov and Shabat [7]. Auto-Bäcklund transformations for the nonlinear equations associated with the commutativity condition (1.2) are readily generated thereby. Thus, it is required that the gauge transformations preserve the form of the operators L_1 and L_2 so that, since

$$[L'_1, L'_2] = G[L_1, L_2]G^{-1} = 0 \quad , \tag{1.4}$$

it follows that the functions $u'_1,..,u'_n$ obey the same nonlinear system as the functions $u_1,..,u_n$. In order to extract explicit transformations it is necessary to assume specific forms for the gauge operator G. Thus, if one selects the operator G as an appropriate Volterra-type integral operator then one obtains the celebrated Gelfand-Levitan-Marchenko linear integral equations and the corresponding reconstruction formulae for their potentials. The details of this approach to the IST are set down in [2]. The method may be linked to a development of the classical Darboux method introduced in 1882 which generates solutions to related Schrödinger equations (Darboux [8]). The connection has been described by Levi, Ragnisco and Sym [9]. Therein, the analogue of the classical Darboux transformation for the Zakharov-Shabat-AKNS spectral scheme was presented.

Extension to 2+1 dimensions of the established methods developed for nonlinear integrable equations in 1+1 dimensions is a subject of current research. Auto-Bäcklund transformations generated via gauge invariance have recently been shown to play a key role in this area. Thus, until recently, the only localized objects in 2+1 dimensions were the so-called lump solitons which decay algebraically at infinity. However, in an important recent development, Bäcklund transformations have been exploited by Boiti et al [10] to derive truly two-dimensional coherent structures of the Davey-Stewartson equation which decay exponentially in all directions. The associated nonlinear superposition principle was used to study the scattering properties of these localized phenomena which have come to known as dromions [11]. The 2+1-dimensional Bäcklund transformations employed were constructed by the so-called Dressing Method based on appropriate gauge transformations (Boiti, Konopelchenko and Pempinelli [12]). This dressing approach had been earlier used in 2+1 dimensions by Levi, Pilloni and Santini [13] to construct auto-Bäcklund transformations for the Kadomtsev-Petviashvili (KP), two-dimensional three-wave and Davey-Stewartson equation in turn. The latter represents a 2+1-dimensional integrable extension of the nonlinear Schrödinger equation wherein the spatial variables occur on an equal footing. The auto-Bäcklund transformations of these systems may be derived from generalized Darboux theorems for the underlying scattering problems. Solutions may be represented in terms of eigenfunctions solving these linear equations [14]. Generalized Darboux theorems of Moutard-type have recently been used by Athorne and Nimmo [15] to construct dromion solutions of the Nizhnik-Novikov-Veselov equation [16-18]. This, in turn, represents a 2+1-dimensional extension of the Korteweg-de Vries equation in which the spatial variables arise in a symmetric manner. The corresponding 2+1-dimensional sine Gordon equation has recently been introduced by Konopelchenko and Rogers [19].

Gauge transformations, in conjunction with reciprocal transformations, also play a key role in links between scattering schemes associated with classes of nonlinear integrable systems. Reciprocal transformations have been extensively employed in Continuum Mechanics not only to reveal hidden symmetries in nonlinear systems but also to solve certain nonlinear boundary value problems of practical interest. Such applications are described, in detail, in Rogers and Shadwick [20] and Rogers and Ames [21]. In the present context of soliton theory, reciprocal transformations may be shown to be a key component in the link between the AKNS and WKI inverse scattering schemes [22]. Moreover, it was shown in [22] that the Dym equation [2] is invariant under a reciprocal transformation. This invariance is readily extended to the Dym hierarchy as set down by Calogero and Degasperis [5]. This result may be shown to induce a generic auto-Bäcklund transformation for the Korteweg-de Vries hierarchy. A reciprocal connection between the KP equation and the 2+1-dimensional Dym equation as introduced by Konopelchenko and Dubrovsky [24] was noted in [25]. The present paper is devoted to a general discussion of gauge and reciprocal links in 2+1 dimensions. A background account of the subject is to be found in a recent review article by Konopelchenko and Rogers [26].

The classes of integrable hierarchies discussed here form a generalization of the KP hierarchy as constructed in Sato theory [27-29]. The KP hierarchy is associated with generalized Lax equations of the type

$$L_{t_q} = [(L^q)_+, L] \quad , \quad q \in \mathbb{N} \; , \tag{1.5}$$

where L is a pseudo-differential operator and $(..)_+$ is the projection to the purely differential part of the operator. In [30] Adler gave a Lie-algebraic interpretation for such equations, embedding integrable partial differential equations such as the Korteweg-de Vries equation in a general Lie-algebraic setting, now known as the Adler-Kostant-Symes (AKS) scheme. The AKS approach turned out to be a special case of an even more general algebraic structure involving classical r-matrices. Following Drinfeld's ideas [31], Semenov showed how multi-Hamiltonian structures and dressing methods for integrable equations originate from general algebraic and group theoretic structures [32,33]. The AKS scheme is associated with special r-matrices which stem from decompositions of the underlying Lie algebra. This general approach shows that the Lax equations (1.5) will be integrable for any projection $(..)_+$ associated with a Lie-algebraic decomposition of a space of so-called pseudo-differential symbols. It turns out that there are, in fact, three natural decompositions, leading in turn to three classes of Lax equations (1.5), corresponding to different realizations of the projection $(..)_+$. Such modifications of the original Lax equation were noted by Reiman [34], and a systematic analysis was performed by Kupershmidt [35]. These modified hierarchies have also been recorded recently by Kiso [36] in the context of Sato equations (1.5).

[2] It has been recently noted that the Dym equation arises in the analysis of the Saffman-Taylor-problem with surface tension [23]. The authors are indebted to Dr. Ockendon and Dr. Howieson of Oxford University for drawing the attention of the authors to this work.

In this paper, we give an account of invariances and novel relations between the three natural Lax hierarchies of type (1.5). In Section 2, these hierarchies are introduced and shown to be associated with KP, modified KP and Dym hierarchies in 2+1 dimensions. In Section 3, a general theory of gauge transformations and reciprocal links for these hierarchies is developed, which reveals the intimate connections between the different hierarchies. These links are formulated as a collection of Darboux theorems for the associated linear problems. In Section 4, Bäcklund transformations for the 2+1-dimensional integrable equations associated with the Lax hierarchies are extracted from the general statements of Section 3. In Section 5, results are presented on the commutativity of the corresponding Bäcklund diagrams.

The general results of Section 3 are not restricted to 2+1-dimensional integrable equations. Thus, the Lax hierarchies may be shown by standard reduction techniques to encapsulate a diversity of 1+1-dimensional integrable hierarchies. Again, Bäcklund transformations and reciprocal links emerge from the general results presented herein. A detailed account of these reductions is presented in [37]. The results of this paper provide a unified approach to gauge transformations and reciprocal links for a broad class of integrable nonlinear equations in 1+1 and 2+1 dimensions associated with generalized Lax equations of the type (1.5).

2 Pseudo-differential operators and integrable hierarchies in 2+1 dimensions

We consider the algebra

$$g = \{ \sum_{i \ll \infty} u_i(x) \partial_x^i \} \tag{2.1}$$

of pseudo-differential operators of a "space" variable x with coefficients u_i being functions of x. For positive powers i of the differential operator ∂_x, the algebraic multiplication with a multiplication operator –represented by a function $u = u(x)$– is given by the usual Leibnitz rule

$$\partial_x^i u = \sum_{j \geq 0} \begin{pmatrix} i \\ j \end{pmatrix} \frac{\partial^j u}{\partial x^j} \, \partial_x^{i-j} \quad . \tag{2.2}$$

With the same definition for negative powers, so that in particular $\partial_x^{-1} u = u\partial_x^{-1} - u_x \partial_x^{-2} + u_{xx}\partial_x^{-3} - \ldots$, we obtain a well defined associative algebraic structure on g.

As observed in [34,35] there is a straightforward construction of integrable hierarchies of Lax equations on the space g induced by decompositions

$$g = g_{\geq k} \oplus g_{<k} = \{ \sum_{i \geq k} u_i \partial_x^i \} \oplus \{ \sum_{i<k} u_i \partial_x^i \} \tag{2.3}$$

It is clear that, for any non-negative integer k, the subspace $g_{\geq k}$ provides a subalgebra of purely differential operators in g. It is also obvious that, for any negative k, the subspace $g_{<k}$ yields a subalgebra of integration operators. Further Lie subalgebras in g are given by the subspaces $g_0 = \{ u_0 \}$ of multiplication operators and $g_{0,1} = \{ u_0 + u_1 \partial_x \}$ of first order differential operators. It is readily checked that $g_{<1} = g_{<0} \oplus g_0$ and $g_{<2} = g_{<0} \oplus g_{0,1}$ again form closed Lie subalgebras in g. Thus, the three cases $k = 0, 1, 2$ are distinguished by the fact that $g = g_{\leq k} \oplus g_{>k}$ establishes a Lie algebra decomposition of g and hence the celebrated Adler-Kostant-Symes (AKS) scheme [30] can be invoked to construct hierarchies of commuting Lax equations on g. In a slightly more general context, for $k = 0, 1, 2$ the linear maps $r_k = P_{\geq k} - P_{<k}$ satisfy the modified Yang-Baxter

equation [32] on g and thus allow a straightforward approach to the Hamiltonian structure of the integrable equations constructed in the AKS scheme. Here,

$$P_{\geq k}(\sum_i u_i\partial_x^i) = \sum_{i\geq k} u_i\partial_x^i \quad , \quad P_{<k}(\sum_i u_i\partial_x^i) = \sum_{i<k} u_i\partial_x^i \tag{2.4}$$

are the natural projections to the Lie subalgebras $g_{\geq k}$ and $g_{<k}$. In the following, no use will be made of the underlying Hamiltonian structures inherent in the nonlinear systems considered. A detailed discussion of such Hamiltonian aspects is given in [37,38].

The AKS approach immediately leads to the particular result that for fixed $k = 0, 1$ or 2 the equations

$$L_{t_q} = [P_{\geq k}(L^q), L] \quad , \quad q = 1, 2, 3, \ldots , \tag{2.5}$$

commute. Herein, the power q labels the hierarchy of equations associated with a given Lax operator $L \in g$. By means of the identity $[P_{\geq k}(L^q), L] = -[P_{<k}(L^q), L]$ it is readily seen that, for the cases $k = 0, 1, 2$, the right hand side of (2.5) yields a pseudo-differential operator of the same "highest" order as L. Specifically, if L is given by $L = \sum_{i\leq N} u_i\partial_x^i$, then $[P_{\geq k}(L^q), L] = \sum_{i\leq N+k-2} b_i\partial_x^i$ with appropriate coefficients b_i. Thus, for $k = 0$ the two highest coefficients u_N and u_{N-1} in L will not inherit any dynamics from the equations (2.5) and hence can be chosen arbitrarily. For $k = 1$ only the highest coefficient u_N may be chosen, for $k = 2$ all coefficients in L are dynamical variables.
We adopt an extension of the standard construction of the KP hierarchy via Sato theory (see [29] for an elementary introduction to the subject and further references). Thus, we choose the operator

$$k = 0: \quad L_{KP} \quad = \quad \partial_x + \tfrac{1}{2}u\partial_x^{-1} + u_2\partial_x^{-2} + u_3\partial_x^{-3} + \ldots \quad , \tag{2.6}$$

as in standard Sato theory, together with

$$\begin{aligned} k = 1: \quad L_{mKP} \quad &= \quad \partial_x + v + v_1\partial_x^{-1} + v_2\partial_x^{-2} + v_3\partial_x^{-3} + \ldots \quad , \\ k = 2: \quad L_D \quad &= \quad r\partial_x + r_0 + r_1\partial_x^{-1} + r_2\partial_x^{-2} + r_3\partial_x^{-3} + \ldots \quad . \end{aligned} \tag{2.7}$$

Here the "highest coefficients" u,v and r carry no index, since they will be taken as distinguished fields which satisfy identifiable integrable equations. These are the KP hierarchy for $k = 0$, the modified KP hierarchy for $k = 1$ and a 2+1-dimensional Dym hierarchy for $k = 2$, as indicated by the choice of subscripts for the Lax operators. The remaining fields $u_2, u_3, \ldots, v_1, v_2, \ldots, r_0, r_1, \ldots$ may be regarded as auxiliary fields to be eliminated in the construction.

2.1 The Kadomtsev-Petviashvili hierarchy

Here, we review the construction of the KP hierarchy via Sato theory. Thus, with $L = L_{KP} = \partial_x + \frac{1}{2}u\partial_x^{-1} + u_2\partial_x^{-2} + u_3\partial_x^{-3} + \ldots$ one readily calculates the differential operators

$$\begin{aligned} P_{\geq 0}(L) \; &= \partial_x \quad , \\ P_{\geq 0}(L^2) &= \partial_x^2 + u \quad , \\ P_{\geq 0}(L^3) &= \partial_x^3 + \tfrac{3}{2}u\partial_x + \tfrac{3}{2}u_x + 3u_2 \quad . \end{aligned} \tag{2.8}$$

In general, it is clear that the operator $P_{\geq 0}(L^q)$ will be a differential operator of order q involving the fields u, u_2,u_3,...u_{q-1}. Thus, the first of the evolution equations (2.5) give

$$L_{t_1} = [P_{\geq 0}(L), L] \quad \leftrightarrow \quad \begin{cases} u_{t_1} \; = u_x \quad , \\ u_{2t_1} = u_{2x} \quad , \\ u_{3t_1} = u_{3x} \quad , \\ \quad \ldots \quad , \end{cases} \tag{2.9}$$

$$L_{t_2} = [P_{\geq 0}(L^2), L] \quad \leftrightarrow \quad \begin{cases} u_y \ = u_{xx} \ + 4u_{2x} \quad , \\ u_{2y} = u_{2xx} + 2u_{3x} + \frac{1}{2}uu_x \quad , \\ u_{3y} = u_{3xx} + 2u_{4x} - \frac{1}{2}uu_{xx} + 2u_2u_x \quad , \\ \quad \dots \quad , \end{cases} \tag{2.10}$$

$$L_{t_3} = [P_{\geq 0}(L^3), L] \quad \leftrightarrow \quad \begin{cases} u_t \ = u_{xxx} \ + 3uu_x + 6u_{2xx} + 6u_{3x} \quad , \\ u_{2t} = u_{2xxx} + 3(uu_2)_x + 3u_{3xx} + 3u_{4x} \quad , \\ \quad \dots \quad , \end{cases} \tag{2.11}$$

with the more convenient notation $y = t_2$, $t = t_3$. These are the first equations of a countable hierarchy of commuting coupled equations involving all the fields u, u_2, u_3,.. . For each q, the corresponding evolution equation represents a 1+1-dimensional system in the two independent variables t_q and x and infinitely many dependent fields. Certain procedures are available whereby this system may be reduced to hierarchies of 1+1-dimensional equations which involve just a finite number of these fields ("N-reduction" [29]).

Another way of obtaining commuting equations which involve one field only makes use of the particular structure of the equations associated with $t_2 = y$. In this connection, it is noted that by a simple integration with respect to x the fields u_2, u_3,..., can be recursively expressed in terms of the field u and its derivatives with respect to y. Hence u_2, u_3,... may be considered as auxiliary fields to be eliminated via (2.10) from the higher equations in the above hierarchy. For example, eliminating u_2 and u_3 via (2.10), one can rewrite the first component of (2.11) in terms of u, its x- and its y-derivatives. The equation thus encountered is the Kadomtsev-Petviashvili (KP) equation

$$4u_{tx} = (u_{xxx} + 6uu_x)_x + 3u_{yy} \quad . \tag{2.12}$$

Elimination of the auxiliary fields from the other components of (2.11) just gives differential consequences of (2.12).

One may also consider the higher equations for u arising from $L_{t_q} = [P_{\geq 0}(L^q), L]$. These are of the form $u_{t_q} = E_q(u, u_2, u_3, ..., u_q)$ with E_q a differential expression of the indicated arguments. After elimination of the auxiliary fields via (2.10) there remains an equation for u only. Since it has to commute with (2.12), it represents the qth member of the KP hierarchy.

2.2 The modified Kadomtsev-Petviashvili hierarchy

A similar analysis, now with $k = 1$ and $L = L_{mKP} = \partial_x + v + v_1\partial_x^{-1} + v_2\partial_x^{-2} + \dots$, leads to the modified KP hierarchy for the field v. In this case,

$$\begin{aligned} P_{\geq 1}(L) \ &= \partial_x \quad , \\ P_{\geq 1}(L^2) &= \partial_x^2 + 2v\partial_x \quad , \\ P_{\geq 1}(L^3) &= \partial_x^3 + 3v\partial_x^2 + 3(v_x + v^2 + v_1)\partial_x \quad . \end{aligned} \tag{2.13}$$

In general, the operator $P_{\geq 1}(L^q)$ will be a differential operator of order q which involves the fields v, v_1,v_2,...v_{q-2}. The first of the evolution equations (2.5) yield

$$L_{t_1} = [P_{\geq 1}(L), L] \quad \leftrightarrow \quad \begin{cases} v_{t_1} \ = v_x \quad , \\ v_{1t_1} = v_{1x} \quad , \\ v_{2t_1} = v_{2x} \quad , \\ \quad \dots \quad , \end{cases} \tag{2.14}$$

$$L_{t_2} = [P_{\geq 1}(L^2), L] \quad \leftrightarrow \quad \begin{cases} v_y &= v_{xx} + 2vv_x + 2v_{1x} \quad , \\ v_{1y} &= v_{1xx} + 2(vv_1)_x + 2v_{2x} \quad , \\ v_{2y} &= v_{2xx} - 2v_1 v_{xx} + 2vv_{2x} + 4v_x v_2 + 2v_{3x} \quad , \\ & \dots \quad , \end{cases} \tag{2.15}$$

$$L_{t_3} = [P_{\geq 1}(L^3), L] \quad \leftrightarrow \quad \begin{cases} v_t &= v_{xxx} + 3v_{1xx} + 3v_{2x} + 3(vv_x)_x + 3v^2 v_x + 6(vv_1)_x \quad , \\ & \dots \quad , \end{cases} \tag{2.16}$$

where again we have set $y = t_2$, $t = t_3$. Now the auxiliary fields v_1, v_2, .. , can be eliminated by expressing them in terms of the field v and its $y-$derivatives. For example, eliminating v_1 and v_2 via (2.15), one can rewrite the first component of (2.16) in terms of v. The equation so obtained is the modified Kadomtsev-Petviashvili (mKP) equation

$$4v_{tx} = (v_{xxx} - 6v^2 v_x)_x + 3v_{yy} + 6v_x v_y + 6v_{xx} D_x^{-1} v_y \quad . \tag{2.17}$$

Here, $D_x^{-1} = \int^x d\xi$ (to be distinguished from the pseudo-differential symbol ∂_x^{-1}) is an integration operator arising from the elimination of the auxiliary fields.

On consideration of the higher equations for v arising from $L_{t_q} = [P_{\geq 1}(L^q), L]$ with $q = 4, 5, ...$, one finds $v_{t_q} = E_q(v, v_1, v_2, ..., v_{q-1})$ where the expression E_q involves derivatives of the indicated arguments. After elimination of the auxiliary fields via (2.15) an equation for v only remains. This represents the qth member of the modified KP hierarchy.

2.3 The Dym hierarchy in 2+1 dimensions

A similar analysis, but now with $k = 2$ and $L = L_D = r\partial_x + r_0 + r_1\partial_x^{-1} + r_2\partial_x^{-2} + ...$, leads to a hierarchy of 2+1-dimensional Dym equations for the field r. Here

$$\begin{aligned} P_{\geq 2}(L) &= 0 \quad , \\ P_{\geq 2}(L^2) &= r^2 \partial_x^2 \quad , \\ P_{\geq 2}(L^3) &= r^3 \partial_x^3 + 3r^2(r_x + r_0)\partial_x^2 \quad . \end{aligned} \tag{2.18}$$

In general, the operator $P_{\geq 2}(L^q)$ will be a differential operator of order q involving the fields r, r_0,r_1,...r_{q-3}. The first of the evolution equations (2.5) provide

$$L_{t_1} = [P_{\geq 2}(L), L] \quad \leftrightarrow \quad \begin{cases} r_{t_1} &= 0 \quad , \\ r_{0t_1} &= 0 \quad , \\ & \dots \quad , \end{cases} \tag{2.19}$$

$$L_{t_2} = [P_{\geq 2}(L^2), L] \quad \leftrightarrow \quad \begin{cases} r_y &= r^2 r_{xx} + 2r^2 r_{0x} \quad , \\ r_{0y} &= r^2 r_{0xx} + 2r(rr_1)_x \quad , \\ r_{1y} &= r^2 r_{1xx} + 2(r^2 r_2)_x - 2r_1(rr_x)_x \quad , \\ & \dots \quad , \end{cases} \tag{2.20}$$

$$L_{t_3} = [P_{\geq 2}(L^3), L] \quad \leftrightarrow \quad \begin{cases} r_t &= r^2 \left(\tfrac{1}{2}(r^2)_{xx} + 3(rr_0)_x + 3r_0^2 + 3rr_1 \right)_x \quad , \\ & \dots \quad , \end{cases} \tag{2.21}$$

where again $y = t_2$, $t = t_3$. Now the auxiliary fields r_0, r_1, ..., can be eliminated by expressing them in terms of the field r and its $y-$derivatives. For instance, eliminating r_0 and r_1 via (2.20), one can rewrite the first component of (2.21) in terms of r. The equation so derived is a 2+1-dimensional version of the Dym equation first constructed by Konopelchenko and Dubrovsky in [24]:

$$4r_t = r^3 r_{xxx} - 3\,\frac{1}{r}\,(r^2 D_x^{-1}(\frac{1}{r})_y)_y \quad . \tag{2.22}$$

Again, the integration $D_x^{-1} = \int^x d\xi$ arises from the elimination of the auxiliary fields.

As before, considering the higher equations for r arising from $L_{t_q} = [P_{\geq 2}(L^q), L]$ with $q = 4, 5, ...$, one finds $r_{t_q} = E_q(r, r_0, r_1, ..., r_{q-2})$ with expressions E_q involving the derivatives of the indicated arguments. After elimination of the auxiliary fields by (2.20) there remains an equation for r only. This represents the qth member of the 2+1-dimensional Dym hierarchy.

3 Gauge transformations and reciprocal links

With the hierarchies of KP, modified KP and Dym equations now formulated via $L_{t_q} = [P_{\geq k}(L^q), L]$ with $k = 0, 1, 2$, it is natural to investigate their links in terms of the pseudo-differential operator L. Since the various equations essentially arise by choice of different k's, it follows that the relations between the differential operators $P_{\geq k}(L^q)$ for different k's require examination. Here, we will obtain general results on such links for the Lax equations (2.5). The resulting transformations for the 2+1-dimensional integrable equations (2.12), (2.17) and (2.22) then are obtained as immediate consequences in Section 4.

3.1 Gauge transformations

Relations between the classes $k = 0$ and $k = 1$ were observed in [35], while a systematic analysis is given in [37]. It turns out that various gauge transformations of the type $L \to \Phi^{-1} L \Phi$, $L \to \Phi \partial_x \Phi^{-1} L \Phi \partial_x^{-1} \Phi^{-1}$ and $L \to \Phi_x^{-1} \partial_x L \partial_x^{-1} \Phi_x$ need to be considered to establish such links. Here Φ is a given function, to be interpreted as a multiplication operator, while Φ^{-1} and Φ_x^{-1} indicate multiplication by $1/\Phi$ and $1/\Phi_x$, respectively. We will show how Miura- and auto-Bäcklund transformations emerge from these considerations in a remarkably simple way. The results originate from the following basic identities on pseudo-differential operators:

Lemma 1:

For arbitrary pseudo-differential operator $A \in g$ one has the following operator identities:

i) $P_{\geq 1}(\Phi^{-1} A \Phi) = \Phi^{-1} P_{\geq 0}(A) \Phi - \Phi^{-1} [\![P_{\geq 0}(A)\Phi]\!]$,

ii) $P_{\geq 1}(\Phi^{-1} A \Phi) = \Phi^{-1} P_{\geq 1}(A) \Phi - \Phi^{-1} [\![P_{\geq 1}(A)\Phi]\!]$,

iii) $P_{\geq 0}(\Phi \partial_x \Phi^{-1} A \Phi \partial_x^{-1} \Phi^{-1}) = \Phi \partial_x \Phi^{-1} P_{\geq 0}(A) \Phi \partial_x^{-1} \Phi^{-1} - \Phi(\Phi^{-1} [\![P_{\geq 0}(A)\Phi]\!])_x \partial_x^{-1} \Phi^{-1}$,

iv) $P_{\geq 1}(\Phi_x^{-1} \partial_x A \partial_x^{-1} \Phi_x) = \Phi_x^{-1} \partial_x P_{\geq 1}(A) \partial_x^{-1} \Phi_x - \Phi_x^{-1} [\![P_{\geq 1}(A)\Phi]\!]_x$.

Here we have used the notation $[\![b]\!]$ to indicate the multiplication operator by the function b. Hence, the symbol $[\![P_{\geq 0}(A)\Phi]\!]$, say, is to denote the multiplication with the function obtained by the differential operator $P_{\geq 0}(A)$ acting on Φ. We note that with this interpretation there is a minor inconsistency in our notation, as we we always regard $P_{\geq 0}(A)\Phi$ as an operator (multiplication with Φ followed by $P_{\geq 0}(A)$). However, the function obtained from the action of $P_{\geq 0}(A)$ on Φ is just the zero-order term of the operator $P_{\geq 0}(A)\Phi$, so that for any differential operator $B = b_0 + b_1 \partial + ...$ we may regard the symbol $[\![B]\!]$ as the projection to its zero-order term: $[\![B]\!] = b_0$.

The identities of Lemma 1 are readily proved. We just indicate the line of argument for i). Thus, first note that only the purely differential part of A contributes to all the terms under consideration. Hence, $P_{\geq 1}(\Phi^{-1}A\Phi) = P_{\geq 1}(\Phi^{-1}P_{\geq 0}(A)\Phi) = P_{\geq 0}(\Phi^{-1}P_{\geq 0}(A)\Phi) - P_0(\Phi^{-1}P_{\geq 0}(A)\Phi)$. Here, $P_0 = [\![.]\!]$ is the projection to the zero-order of the differential operator. With $P_{\geq 0}(\Phi^{-1}P_{\geq 0}(A)\Phi) = \Phi^{-1}P_{\geq 0}(A)\Phi$ and $P_0(\Phi^{-1}P_{\geq 0}(A)\Phi) = \Phi^{-1}[\![P_{\geq 0}(A)\Phi]\!]$ the first statement is evident.

We will see that the above identities lead to gauge links between the integrable equations (2.5) for $k=0$ and $k=1$. Let L be a given operator satisfying a Lax equation (2.5). For a given function $\Phi(x,t_q)$ we consider the gauge-transformed operators $\tilde{L} = \Phi^{-1}L\Phi$, $\tilde{L} = \Phi\partial_x\Phi^{-1}L\Phi\partial_x^{-1}\Phi^{-1}$ and $\tilde{L} = \Phi_x^{-1}\partial_x L\partial_x^{-1}\Phi_x$, in turn, and investigate the dynamics of these new operators. In particular, the question arises, as to what dynamical behavior of the gauge function Φ the new operators will again satisfy a Lax equation of the type (2.5). The answer can be obtained from the following identities:

Lemma 2:

Let $L \in g$ be an arbitrary pseudo-differential operator and let $\Phi(x,t_q) \neq 0$, $\Psi(x,t_q)$ be arbitrary functions. The following identities hold:

i) If $\tilde{L} = \Phi^{-1}L\Phi$ and $\tilde{\Psi} = \Phi^{-1}\Psi$ then

$$\begin{aligned} \tilde{L}_{t_q} - [P_{\geq 1}(\tilde{L}^q), \tilde{L}] &\equiv \Phi^{-1} \left(L_{t_q} - [P_{\geq 0}(L^q), L] \right) \Phi - [\Phi^{-1}[\![\Phi_{t_q} - P_{\geq 0}(L^q)\Phi]\!], \tilde{L}] \ , \\ [\![\tilde{\Psi}_{t_q} - P_{\geq 1}(\tilde{L}^q)\tilde{\Psi}]\!] &\equiv -\Phi^{-2}\Psi \, [\![\Phi_{t_q} - P_{\geq 0}(L^q)\Phi]\!] + \Phi^{-1} \, [\![\Psi_{t_q} - P_{\geq 0}(L^q)\Psi]\!] \ . \end{aligned} \tag{3.1}$$

ii) If $\tilde{L} = \Phi^{-1}L\Phi$ and $\tilde{\Psi} = \Phi^{-1}\Psi$ then

$$\begin{aligned} \tilde{L}_{t_q} - [P_{\geq 1}(\tilde{L}^q), \tilde{L}] &\equiv \Phi^{-1} \left(L_{t_q} - [P_{\geq 1}(L^q), L] \right) \Phi - [\Phi^{-1}[\![\Phi_{t_q} - P_{\geq 1}(L^q)\Phi]\!], \tilde{L}] \ , \\ [\![\tilde{\Psi}_{t_q} - P_{\geq 1}(\tilde{L}^q)\tilde{\Psi}]\!] &\equiv -\Phi^{-2}\Psi \, [\![\Phi_{t_q} - P_{\geq 1}(L^q)\Phi]\!] + \Phi^{-1}[\![\Psi_{t_q} - P_{\geq 1}(L^q)\Psi]\!] \ . \end{aligned} \tag{3.2}$$

iii) If $\tilde{L} = \Phi\partial_x\Phi^{-1}L\Phi\partial_x^{-1}\Phi^{-1}$ and $\tilde{\Psi} = \Phi(\Phi^{-1}\Psi)_x$ then

$$\begin{aligned} \tilde{L}_{t_q} - [P_{\geq 0}(\tilde{L}^q), \tilde{L}] &\equiv \Phi\partial_x\Phi^{-1} \left(L_{t_q} - [P_{\geq 0}(L^q), L] \right) \Phi\partial_x^{-1}\Phi^{-1} \\ &\qquad - [\Phi(\Phi^{-1}[\![\Phi_{t_q} - P_{\geq 0}(L^q)\Phi]\!])_x\partial_x^{-1}\Phi^{-1}, \tilde{L}] \ , \\ [\![\tilde{\Psi}_{t_q} - P_{\geq 0}(\tilde{L}^q)\tilde{\Psi}]\!] &\equiv -\Psi(\Phi^{-1}[\![\Phi_{t_q} - P_{\geq 0}(L^q)\Phi]\!])_x + \Phi(\Phi^{-1}[\![\Psi_{t_q} - P_{\geq 0}(L^q)\Psi]\!])_x \ . \end{aligned} \tag{3.3}$$

iv) If $\tilde{L} = \Phi_x^{-1}\partial_x L\partial_x^{-1}\Phi_x$ and $\tilde{\Psi} = \Phi_x^{-1}\Psi_x$ then

$$\begin{aligned} \tilde{L}_{t_q} - [P_{\geq 1}(\tilde{L}^q), \tilde{L}] &\equiv \Phi_x^{-1}\partial_x \left(L_{t_q} - [P_{\geq 1}(L^q), L] \right) \partial_x^{-1}\Phi_x - [\Phi_x^{-1}[\![\Phi_{t_q} - P_{\geq 1}(L^q)\Phi]\!]_x, \tilde{L}] \ , \\ [\![\tilde{\Psi}_{t_q} - P_{\geq 1}(\tilde{L}^q)\tilde{\Psi}]\!] &\equiv -\Phi_x^{-2}\Psi_x \, [\![\Phi_{t_q} - P_{\geq 1}(L^q)\Phi]\!]_x + \Phi_x^{-1} \, [\![\Psi_{t_q} - P_{\geq 1}(L^q)\Psi]\!]_x \ . \end{aligned} \tag{3.4}$$

The proof of the above is a straightforward consequence of Lemma 1 with $A = L^q$ together with the observation that $\tilde{L}^q = \Phi^{-1}L^q\Phi$ etc. The occurrence of the expressions $\Phi_{t_q} - P_{\geq k}(L^q)\Phi$ in these identities suggests that we consider functions Φ with dynamical behavior given by $\Phi_{t_q} = P_{\geq k}(L^q)\Phi$. Since we consider the iso-spectral dynamics (2.5) for the Lax operators, this is an admissible time-evolution for eigenfunctions Φ of L with time-independent eigenvalues. We fix this notation with:

Definition 1:

For given $k = 0, 1$, or 2, a function $\Phi = \Phi(x, t_q)$ satisfying the linear equations

$$\Phi_{t_q} = P_{\geq k}(L^q)\Phi \quad , \quad q \in \mathbb{N} \ , \tag{3.5}$$

shall be called an **eigenfunction** for the hierarchy of Lax equations $L_{t_q} = [P_{\geq k}(L^q), L]$.

For the cases $k = 0, 1, 2$ under consideration here, these equations are compatible and may be considered simultaneously for different q's. The consequences of Lemma 2 can now be highlighted in the following theorems on gauge transformations of the Lax operators.

Theorem 1: (gauge $k = 0 \ \to k = 1$)

Let $L \in g$ satisfy the hierarchy of Lax equations $L_{t_q} = [P_{\geq 0}(L^q), L]$, $q \in \mathbb{N}$. Let $\Phi \neq 0$ and Ψ be two eigenfunctions of this hierarchy. Then $\tilde{L} = \Phi^{-1} L \Phi$ satisfies the hierarchy $\tilde{L}_{t_q} = [P_{\geq 1}(\tilde{L}^q), \tilde{L}]$ and $\tilde{\Psi} = \Phi^{-1}\Psi$ is an eigenfunction for $\tilde{L}$, that is $\tilde{\Psi}_{t_q} = P_{\geq 1}(\tilde{L}^q)\tilde{\Psi}$.

The proof of the above follows simply from the operator identity given by Lemma 2.i). As demonstrated in Section 2 the Lax equations for L and $\tilde{L}$ define the KP and modified KP hierarchies. Hence the gauge transformation $L \to \tilde{L} = \Phi^{-1} L \Phi$ must give rise to a Miura transform between the KP and its modified hierarchy in terms of eigenfunctions. This is indeed the case, as shall be demonstrated in Section 4. We note that Theorem 1 bears interpretation as a Darboux theorem in the sense that several eigenfunctions of the original operator L can be pulled through the gauge transformation to obtain eigenfunctions of the transformed operator. In a similar fashion the identities of Lemma 2 provide gauge transformations not just between the two different hierarchies described by $k = 0$ and $k = 1$. Importantly, they also induce transformations leaving these equations invariant. For instance, considering eigenfunctions Φ, iii) of Lemma 2 immediately gives rise to an invariance of the KP hierarchy:

Theorem 2: (gauge $k = 0 \ \to \ k = 0$)

Let $L \in g$ satisfy the hierarchy of Lax equations $L_{t_q} = [P_{\geq 0}(L^q), L]$, $q \in \mathbb{N}$. Let $\Phi \neq 0$ and Ψ be two eigenfunctions of this hierarchy. Then $\hat{L} = \Phi \partial_x \Phi^{-1} L \Phi \partial_x^{-1} \Phi^{-1}$ again satisfies the hierarchy $\hat{L}_{t_q} = [P_{\geq 0}(\hat{L}^q), \hat{L}]$ and $\hat{\Psi} = \Phi(\Phi^{-1}\Psi)_x$ is an eigenfunction for $\hat{L}$, that is $\hat{\Psi}_{t_q} = P_{\geq 0}(\hat{L}^q)\hat{\Psi}$.

Parts ii) and iv) of Lemma 2 describe invariances of the modified KP hierarchy:

Theorem 3: (gauge $k = 1 \ \to \ k = 1$)

Let $L \in g$ satisfy the hierarchy of Lax equations $L_{t_q} = [P_{\geq 1}(L^q), L]$, $q \in \mathbb{N}$. Let $\Phi \neq 0$ and Ψ be two eigenfunctions of this hierarchy. Then

$$\begin{aligned}
\text{a)} \quad & \tilde{L} = \Phi^{-1} L \Phi \ , \ \tilde{\Psi} = \Phi^{-1}\Psi \ , \\
\text{or b)} \quad & \tilde{L} = \Phi_x^{-1} \partial_x L \partial_x^{-1} \Phi_x \ , \ \tilde{\Psi} = \Phi_x^{-1}\Psi_x \ , \\
\text{or c)} \quad & \tilde{L} = \Phi_x^{-1} \Phi^2 \partial_x \Phi^{-1} L \Phi \partial_x^{-1} \Phi_x \Phi^{-2} \ , \ \tilde{\Psi} = \Phi_x^{-1}(\Psi_x \Phi - \Psi \Phi_x) \ ,
\end{aligned}$$

satisfy $\tilde{L}_{t_q} = [P_{\geq 1}(\tilde{L}^q), \tilde{L}]$ and $\tilde{\Psi}_{t_q} = P_{\geq 1}(\tilde{L}^q)\tilde{\Psi}$.

Parts a) and b) are simple consequences of parts ii) and iv) of Lemma 2. It should be noted that part c) is a simple composition of the first two transformations. Observe that due to the projections $P_{\geq 1}(L^q)$ the constant function 1 will be an eigenfunction trivially. Hence, starting with L and three eigenfunctions Φ, Ψ and 1, part a) will lead to $\tilde{L} = \Phi^{-1}L\Phi$, equipped, in turn, with the eigenfunctions $\tilde{\Psi} = \Phi^{-1}\Psi$ and Φ^{-1}. Now, applying b) using the eigenfunction Φ^{-1} leads to $\tilde{L} \to \tilde{\tilde{L}} = ((\Phi^{-1})_x)^{-1}\partial_x \tilde{L}\partial_x^{-1}(\Phi^{-1})_x = \Phi_x^{-1}\Phi^2\partial_x\Phi^{-1}L\Phi\partial_x^{-1}\Phi_x\Phi^{-2}$, equipped with an eigenfunction $\tilde{\tilde{\Psi}} = ((\Phi^{-1})_x)^{-1}(\Phi^{-1}\Psi)_x = -\Phi_x^{-1}(\Psi_x\Phi - \Psi\Phi_x)$. This is the result described by c).

Remark 1:

Since any constant function will trivially be an eigenfunction for the hierarchy (2.5) with $k = 1$, we may reformulate part a) of Theorem 3 as the gauge transformation

$$L \to \tilde{L} = (c\Phi + d)^{-1}L(c\Phi + d) \quad , \quad \Psi \to \tilde{\Psi} = (c\Phi + d)^{-1}(a\Psi + b) \tag{3.6}$$

with arbitrary constants a, b, c and d.

It is emphasized that the transformations embodied in Theorems 1-3 are not just a collection of unconnected results on gauge transformations. In fact, all these transformations are connected, and the results can be summarized in the following commuting Darboux diagram:

$$
\begin{array}{ccccc}
\underline{k=0} & & & & \underline{k=1} \\
 & & & \nearrow & \boxed{\begin{array}{rcl} \bar{L}_{t_q} &=& [P_{\geq 1}(\bar{L}^q), \bar{L}] \\ \bar{\Phi}_{t_q} &=& P_{\geq 1}(\bar{L}^q)\bar{\Phi} \end{array}} \\
 & & & & \uparrow\downarrow \\
 & & \begin{array}{rcl} \bar{L} &=& \Psi^{-1}L\Psi \\ \bar{\Phi} &=& \Psi^{-1}\Phi \end{array} & & \begin{array}{c} \tilde{L} = \bar{\Phi}^{-1}\bar{L}\bar{\Phi} \ , \ \bar{L} = \tilde{\Psi}^{-1}\tilde{L}\tilde{\Psi} \\ \bar{\Phi} = \tilde{\Psi}^{-1} \end{array} \\
 & \nearrow & & & \uparrow\downarrow \\
\boxed{\begin{array}{rcl} L_{t_q} &=& [P_{\geq 0}(L^q), L] \\ \Phi_{t_q} &=& P_{\geq 0}(L^q)\Phi \\ \Psi_{t_q} &=& P_{\geq 0}(L^q)\Psi \end{array}} & \to & \begin{array}{rcl} \tilde{L} &=& \Phi^{-1}L\Phi \\ \tilde{\Psi} &=& \Phi^{-1}\Psi \end{array} & \to & \boxed{\begin{array}{rcl} \tilde{L}_{t_q} &=& [P_{\geq 1}(\tilde{L}^q), \tilde{L}] \\ \tilde{\Psi}_{t_q} &=& P_{\geq 1}(\tilde{L}^q)\tilde{\Psi} \end{array}} \\
\downarrow & & & & \downarrow \\
\begin{array}{rcl} \hat{L} &=& \Phi\partial_x\Phi^{-1}L\Phi\partial_x^{-1}\Phi^{-1} \\ \hat{\Psi} &=& \Phi(\Phi^{-1}\Psi)_x \end{array} & & & & L^* = \tilde{\Psi}_x^{-1}\partial_x\tilde{L}\partial_x^{-1}\tilde{\Psi}_x \\
\downarrow & & & & \downarrow \\
\boxed{\begin{array}{rcl} \hat{L}_{t_q} &=& [P_{\geq 0}(\hat{L}^q), \hat{L}] \\ \hat{\Psi}_{t_q} &=& P_{\geq 0}(\hat{L}^q)\hat{\Psi} \end{array}} & \to & L^* = \hat{\Psi}^{-1}L\hat{\Psi} & \to & \boxed{L^*_{t_q} = [P_{\geq 1}(L^{*q}), L^*]}
\end{array}
$$

Fig 1. The Darboux Diagram
The KP and mKP Hierarchies

In the above, we have started with an operator L giving rise to the KP hierarchy corresponding to $k = 0$. If two eigenfunctions Φ and Ψ for L are given, then each may be used to perform the gauge transformation of Theorem 1 to either $\bar{L}$ or $\tilde{L}$. Each of the new operators $\bar{L}$ and $\tilde{L}$ represents a solution of the modified KP hierarchy. In each case the second eigenfunction may be transformed to a new eigenfunction $\bar{\Phi}$ or $\tilde{\Psi}$ for $\bar{L}$ or $\tilde{L}$, respectively. The operators $\bar{L}$ and $\tilde{L}$ are gauge equivalent via Theorem 3.a), where one either may use the new eigenfunction $\bar{\Phi}$ to map $\bar{L}$ to $\tilde{L}$ or one may use $\tilde{\Psi}$ to map $\tilde{L}$ to $\bar{L}$, respectively. In fact, for the gauge transformation $\bar{L} \to \tilde{L} = \bar{\Phi}^{-1}\bar{L}\bar{\Phi}$ the eigenfunction $\tilde{\Psi} = \bar{\Phi}^{-1} = \bar{\Psi}\bar{\Phi}^{-1}$ may be regarded as the image of the trivial eigenfunction $\bar{\Psi} \equiv 1$ for $\bar{L}$ in Theorem 3a). This trivial eigenfunction, in turn, may be regarded as the image $\bar{\Psi} = \Psi\Psi^{-1}$ of the original eigenfunction Ψ when using Theorem 1 to map L to $\bar{L}$. Similar arguments apply for $\tilde{L}$ instead of $\bar{L}$.

One may also use Theorem 2 with the eigenfunction Φ of L to obtain a new operator $\hat{L}$ representing a further solution of the KP, equipped with an eigenfunction $\hat{\Psi}$ which is the image of the second eigenfunction Ψ of L. Theorem 1 with $\hat{\Psi}$ generates a further solution L^* of the modified KP. As shown in Figure 1, this L^* coincides with that obtained from $\tilde{L}$ and $\tilde{\Psi}$ via Theorem 3.b).

Finally, the gauge transformation of Theorem 3c) represents the composition of the two transformations $\bar{L} \to \tilde{L} \to L^*$ via $\bar{\Phi}$ and $\tilde{\Psi} = \bar{\Phi}^{-1}$.

3.2 Reciprocal links

Links between the classes $k = 0$ and $k = 1$ of the system (2.5) have now been established. We next reveal the link to the remaining case $k = 2$. For this we need to take into account a change of the independent variable x.

Definition 2:

With a given function $\Phi(x, t_q)$ we introduce new independent variables $x' = \Phi(x, t_q)$, ${t'}_q = t_q$. To each function $a(x, t_q)$ we associate the "linked" function $a'(x', {t'}_q)$ defined by $a'(x', {t'}_q) = a(x, t_q)$. In the immediate sequel, since the t_q enter merely as parameters in the following Lemma 3, we denote $\Phi(x, t_q)$ by $\Phi(x)$, $a(x, t_q)$ by $a(x)$. Observing that $\partial_x = \Phi_x \partial_{x'}$ we associate the operator $A' = \sum_i a'_i(x')(\Phi_x \partial_{x'})^i$ to a given pseudo-differential operator $A = \sum_i a_i(x)\partial_x^i$. We call the pair $A'(x') = A(x)$ "linked" by the transformation $x' = \Phi(x)$. We denote by $P'_{\geq k}$ the "linked" projection operator $P'_{\geq k}(\sum_i b'_i(x')\partial_{x'}^i) = \sum_{i \geq k} b'_i(x')\partial_{x'}^i$, related to the powers of the new symbol $\partial_{x'}$.

From $\partial_x^n = (\Phi_x \partial_{x'})^n$ one immediately obtains the identity $P'_{\geq 1}(A') \equiv P_{\geq 1}(A)$ for linked operators. Considering $P'_{\geq 1}(A') = \sum_{i \geq 1} b'_i \partial_{x'}^i$ one concludes that the lowest coefficient b'_1 can be obtained by letting this operator act on x', that is $b'_1 = [\![P'_{\geq 1}(A')x']\!]$. With $P'_{\geq 1}(A') = P_{\geq 1}(A)$ and $x' = \Phi$ one has $b'_1 = [\![P_{\geq 1}(A)\Phi]\!]$, that is $P'_{\geq 2}(A') = P_{\geq 1}(A) - [\![P_{\geq 1}(A)\Phi]\!]\partial_{x'}$. All statements of the following lemma may be verified by similar arguments:

Lemma 3:

For any pair of pseudo-differential operators $A'(x') = A(x)$ linked via the transformation $x' = \Phi(x)$ one has the identities

$$\begin{aligned} i) \quad & {P'}_{\geq 2}(A') \equiv P_{\geq 1}(A) - [\![P_{\geq 1}(A)\Phi]\!]\partial_{x'} \\ \text{and} \quad ii) \quad & {P'}_{\geq 2}(A') \equiv P_{\geq 2}(A) - [\![P_{\geq 2}(A)\Phi]\!]\partial_{x'} \quad . \end{aligned}$$

For the transformation $x' = 1/x$ the operator $A'(x') = x^{-1}A(x)x$ satisfies

$$iii) \quad P'_{\geq 2}(A') \equiv x^{-1}P_{\geq 2}(A)x \quad .$$

For the transformation $x' = \Phi_x(x)$ the operator $A'(x') = \partial_x A(x) \partial_x^{-1}$ satisfies

$$iv) \qquad P'_{\geq 2}(A') \equiv \partial_x P_{\geq 2}(A) \partial_x^{-1} - [\![P_{\geq 2}(A)\Phi]\!]_x \partial_{x'} \quad .$$

As before, these identities immediately lead to identities for Lax equations of type (2.5) with $k = 2$. We start with a given function $\Phi(x, t_q)$ of both space and time and introduce the new coordinates $x' = \Phi(x, t_q), t'_q = t_q$. Let $L'(x', t'_q) = L(x, t_q)$ be linked to a given operator $L(x, t_q)$ as defined above. Observing

$$\partial_x = \Phi_x \partial_{x'} \quad , \quad \partial_{t_q} = \partial_{t'_q} + \Phi_{t_q} \partial_{x'} \tag{3.7}$$

one calculates $L'_{t'_q} = [\partial_{t'_q}, L'] = [\partial_{t_q}, L] - [\Phi_{t_q}\partial_{x'}, L'] = L_{t_q} - [\Phi_{t_q}\partial_{x'}, L']$. Using Lemma 3.i) with $A = L^q$, $A' = L'^q$ one finds the identity

$$L'_{t'_q} - [P'_{\geq 2}(L'^q), L'] = L_{t_q} - [P_{\geq 1}(L^q), L] \; - \; [[\![\Phi_{t_q} - P_{\geq 1}(L^q)\Phi]\!] \partial_{x'}, L'] \tag{3.8}$$

for arbitrary operators $L' = L$ and arbitrary functions Φ. The results of the following lemma follow similarly.

Lemma 4:

Let $L = L(x, t_q) \in g$ be an arbitrary pseudo-differential operator. Let $\Phi(x, t_q)$ and $\Psi(x, t_q)$ be arbitrary functions. The following identities hold:

i) If $x' = \Phi(x, t_q)$, $t'_q = t_q$, $L'(x', t'_q) = L(x, t_q)$, $\Psi'(x', t'_q) = \Psi(x, t_q)$ then

$$\begin{aligned} L'_{t'_q} - [P'_{\geq 2}(L'^q), L'] &\equiv L_{t_q} - [P_{\geq 1}(L^q), L] - [[\![\Phi_{t_q} - P_{\geq 1}(L^q)\Phi]\!]\partial_{x'}, L'] \; , \\ [\![\Psi'_{t'_q} - P'_{\geq 2}(L'^q)\Psi']\!] &\equiv [\![\Psi_{t_q} - P_{\geq 1}(L^q)\Psi]\!] - [\![\Phi_{t_q} - P_{\geq 1}(L^q)\Phi]\!]\Psi'_{x'} \; . \end{aligned} \tag{3.9}$$

ii) If $x' = \Phi(x, t_q)$, $t'_q = t_q$, $L'(x', t'_q) = L(x, t_q)$, $\Psi'(x', t'_q) = \Psi(x, t_q)$ then

$$\begin{aligned} L'_{t'_q} - [P'_{\geq 2}(L'^q), L'] &\equiv L_{t_q} - [P_{\geq 2}(L^q), L] - [[\![\Phi_{t_q} - P_{\geq 2}(L^q)\Phi]\!]\partial_{x'}, L'] \; , \\ [\![\Psi'_{t'_q} - P'_{\geq 2}(L'^q)\Psi']\!] &\equiv [\![\Psi_{t_q} - P_{\geq 2}(L^q)\Psi]\!] - [\![\Phi_{t_q} - P_{\geq 2}(L^q)\Phi]\!]\Psi'_{x'} \; . \end{aligned} \tag{3.10}$$

iii) If $x' = x^{-1}$, $t'_q = t_q$, $L'(x', t'_q) = x^{-1}L(x, t_q)x$, $\Psi'(x', t'_q) = x^{-1}\Psi(x, t_q)$ then

$$\begin{aligned} L'_{t'_q} - [P'_{\geq 2}(L'^q), L'] &\equiv x^{-1} \; (\; L_{t_q} - [P_{\geq 2}(L^q), L] \;) \; x \; , \\ [\![\Psi'_{t'_q} - P'_{\geq 2}(L'^q)\Psi']\!] &\equiv x^{-1} \; [\![\Psi_{t_q} - P_{\geq 2}(L^q)\Psi]\!] \; . \end{aligned} \tag{3.11}$$

iv) If $x' = \Phi_x(x, t_q)$, $t'_q = t_q$, $L'(x', t'_q) = \partial_x L(x, t_q)\partial_x^{-1}$, $\Psi'(x', t'_q) = \Psi_x(x, t_q)$ then

$$\begin{aligned} L'_{t'_q} - [P'_{\geq 2}(L'^q), L'] &\equiv \partial_x \; (\; L_{t_q} - [P_{\geq 2}(L^q), L] \;) \; \partial_x^{-1} - [[\![\Phi_{t_q} - P_{\geq 2}(L^q)\Phi]\!]_x \partial_{x'}, L'] \; , \\ [\![\Psi'_{t'_q} - P'_{\geq 2}(L'^q)\Psi']\!] &\equiv [\![\Psi_{t_q} - P_{\geq 2}(L^q)\Psi]\!]_x - [\![\Phi_{t_q} - P_{\geq 2}(L^q)\Phi]\!]_x \Psi'_{x'} \; . \end{aligned} \tag{3.12}$$

These identities lead in a natural way to relations and invariances of the Lax equations (2.5) with $k = 1$ and $k = 2$, when we consider eigenfunctions Φ for L. Thus, i) of Lemma 4 immediately leads to a link between the modified KP hierarchy ($k = 1$) and the Dym hierarchy ($k = 2$) as seen below:

Theorem 4: (link $k=1 \ \to \ k=2$)

Let $L = L(x,t_q)$ satisfy $L_{t_q} = [P_{\geq 1}(L^q), L]$, $q \in \mathbb{N}$. Let $\Phi(x,t_q)$ and $\Psi(x,t_q)$ be two eigenfunctions of this hierarchy. Then $L'(x',t'_q) = L(x,t_q)$, linked by the transformation $x' = \Phi(x,t_q), t'_q = t_q$, satisfies the hierarchy

$$L'_{t'_q} = [P'_{\geq 2}(L'^q), L'] \tag{3.13}$$

and $\Psi'(x',t'_q) = \Psi(x,t_q)$ is an eigenfunction for (3.13), that is $\Psi'_{t'_q} = P'_{\geq 2}(L'^q)\Psi'$.

Similarly, parts ii),iii) and iv) of Lemma 4 describe invariances of the 2+1-dimensional Dym hierarchy:

Theorem 5: (links $k=2 \ \to \ k=2$)

Let $L = L(x,t_q)$ satisfy $L_{t_q} = [P_{\geq 2}(L^q), L]$, $q \in \mathbb{N}$. Let $\Phi(x,t_q)$ and $\Psi(x,t_q)$ be two eigenfunctions of this hierarchy. Consider the following transformations:

$$\begin{array}{lllll}
a) & x' = \Phi(x,t_q) & , \ t'_q = t_q & , \ L'(x',t'_q) = L(x,t_q) & , \ \Psi'(x',t'_q) = \Psi(x,t_q) \ , \\
b) & x' = x^{-1} & , \ t'_q = t_q & , \ L'(x',t'_q) = x^{-1}L(x,t_q)x & , \ \Psi'(x',t'_q) = x^{-1}\Psi(x,t_q) \ , \\
c) & x' = \Phi_x(x,t_q) & , \ t'_q = t_q & , \ L'(x',t'_q) = \partial_x L(x,t_q)\partial_x^{-1} & , \ \Psi'(x',t'_q) = \Psi_x(x,t_q) \ , \\
d) & x' = x - \Phi_x^{-1}\Phi & , \ t'_q = t_q & , \ L'(x',t'_q) = \Phi_x^{-1}\Phi^2\partial_x\Phi^{-1}L(x,t_q)\Phi\partial_x^{-1}\Phi_x\Phi^{-2} \ , & \\
 & & & \qquad \Psi'(x',t'_q) = \Phi_x^{-1}(\Psi_x\Phi - \Psi\Phi_x) \ . &
\end{array}$$

Then, in all cases, L' satisfies $L'_{t'_q} = [P'_{\geq 2}(L'^q), L']$ and Ψ' is an eigenfunction of L' which satisfies $\Psi'_{t'_q} = P'_{\geq 2}(L'^q)\Psi'$.

In particular, the function x is an eigenfunction for L, trivially. Hence, for the case a), the inverse function $x = \Phi^{(-1)}(x',t'_q)$ will always be an eigenfunction for L'. Parts a), b) and c) are trivial consequences of parts ii), iii) and iv) of Lemma 4. Statement d) is a simple composition of the first three transformations. We start with an operator L with an eigenfunction Φ. Introducing $x' = \Phi$ we have the new operator $L' = L$, equipped with the eigenfunction $x = \Phi^{(-1)}(x',t'_q)$. Applying b) we introduce $x'' = x'^{-1} = \Phi^{-1}$ and $L'' = x'^{-1}L'x' = \Phi^{-1}L\Phi$, admitting the eigenfunction $x(x')^{-1} = x\Phi^{-1}$. Finally, using c) with the eigenfunction $x\Phi^{-1}$ we find $x''' = (x\Phi^{-1})_{x''} = ((\Phi^{-1})_x)^{-1}(x\Phi^{-1})_x = x - \Phi_x^{-1}\Phi$ and $L''' = \partial_{x''}L''\partial_{x''}^{-1} = ((\Phi^{-1})_x)^{-1}\partial_x\Phi^{-1}L\Phi\partial_x^{-1}(\Phi^{-1})_x$, which is the result described by part d) of Theorem 5.

Remark 2:

Parts a) and b) of Theorem 5 can be conveniently summarized by introducing a transformation

$$x' = (cx+d)^{-1}(a\Phi + b) \ , \quad t'_q = t_q \ , \quad L'(x',t'_q) = (cx+d)^{-1}L(cx+d) \ , \tag{3.14}$$

where a,b,c and d are arbitrary constants. If Φ is an eigenfunction for L, then L' will again satisfy $L'_{t'_q} = [P'_{\geq 2}(L'^q), L']$ and further eigenfunctions Ψ of L are transformed to eigenfunctions Ψ' of L' by $\Psi'(x',t'_q) = (cx+d)^{-1}\Psi(x,t_q)$. This result may be obtained by applying b) of Theorem 5 first, leading to $x' = x^{-1}, L' = x^{-1}Lx$. The eigenfunction Φ for L is transformed into the eigenfunction $\Phi' = x^{-1}\Phi$ for L', so that application of a) leads to $x'' = x^{-1}\Phi$, $L'' = L' = x^{-1}Lx$. By scaling and shifting of x and Φ the transformation (3.14) is, accordingly, a simple consequence of Theorem 5. In particular, choosing $\Phi(x,t_q) = x$, one finds a Möbius transformation $x' = (cx+d)^{-1}(ax+b)$ giving rise to an invariance of the Dym hierarchy (cf. Rogers [25]).

The gauge transformations of Theorem 3 and the links of Theorem 5 are in fact related. We start with an operator L satisfying (2.5) with $k = 1$, giving rise to the modified KP hierarchy. By any eigenfunction Φ of L we can obtain either a solution of the Dym equation by introducing $x' = \Phi, L' = L$, (Theorem 4) or two other solutions of the modified KP hierarchy given by the gauge-transformed operators $\tilde{L} = (c\Phi + d)^{-1}L(c\Phi + d)$ (Theorem 3a),boosted in accordance with Remark 1) and $\hat{L} = \Phi_x^{-1}\partial_x L \partial_x^{-1}\Phi_x$ (Theorem 3b)). According to Theorem 4, further solutions of the Dym equation are obtained, when eigenfunctions $\tilde{\Psi}$ and $\hat{\Psi}$ of $\tilde{L}$ and $\hat{L}$ are given and the transformations $(x, \tilde{L}) \to (\tilde{x}' = \tilde{\Psi}, \tilde{L}' = \tilde{L})$, $(x, \hat{L}) \to (\hat{x}' = \hat{\Psi}, \hat{L}' = \hat{L})$ are performed. We may think of such eigenfunctions $\tilde{\Psi}$ and $\hat{\Psi}$ as being generated from a further eigenfunction Ψ of the original L, that is we may use $\tilde{\Psi} = (c\Phi + d)^{-1}(a\Psi + b)$ (Remark 1) and $\hat{\Psi} = \Phi_x^{-1}\Psi_x$ (Theorem 3b)) as eigenfunctions of $\tilde{L}$ and $\hat{L}$. They lead to further solutions of the Dym equation by the indicated change of variables $x \to \tilde{x}' = \tilde{\Psi}$ and $x \to \hat{x}' = \hat{\Psi}$.

On the other hand, we may define an eigenfunction $\Psi' = \Psi$ for the Dym solution given by the operator $L' = L$ obtained from the first eigenfunction Φ. Now, Theorem 5a), Remark 2 and Theorem 5b) allow us to determine further Dym solutions using Ψ'. It is easily checked that this will lead to the same results as the use of $\tilde{\Psi}$ and $\hat{\Psi}$, so that we have the following commuting Bäcklund diagram:

$$
\begin{array}{ccccc}
\underline{k=1} & & & & \underline{k=2} \\[1em]
\boxed{\begin{array}{c} \tilde{L}_{t_q} = [P_{\geq 1}(\tilde{L}^q), \tilde{L}] \\ \tilde{\Psi}_{t_q} = P_{\geq 1}(\tilde{L}^q)\tilde{\Psi} \end{array}} & \to & \begin{array}{c} \tilde{x}' = \tilde{\Psi} \\ \tilde{t}'_q = t_q, \ \tilde{L}' = \tilde{L} \end{array} & \to & \boxed{\tilde{L}'_{\tilde{t}'_q} = [\tilde{P}'_{\geq 2}(\tilde{L}^q), \tilde{L}']} \\
\uparrow & & & & \uparrow \\
\begin{array}{c} \tilde{L} = (c\Phi + d)^{-1}L(c\Phi + d) \\ \tilde{\Psi} = (c\Phi + d)^{-1}(a\Psi + b) \end{array} & & & & \begin{array}{c} \tilde{L}' = (cx' + d)^{-1}L'(cx' + d) \\ \tilde{x}' = (cx' + d)(a\Psi' + b) \\ \tilde{t}'_q = t'_q \end{array} \\
\uparrow & & & & \uparrow \\
\boxed{\begin{array}{c} L_{t_q} = [P_{\geq 1}(L^q), L] \\ \Phi_{t_q} = P_{\geq 1}(L^q)\Phi \\ \Psi_{t_q} = P_{\geq 1}(L^q)\Psi \end{array}} & \to & \begin{array}{c} x' = \Phi, \ t'_q = t_q, \ L' = L \\ \Psi'(x', t'_q) = \Psi(x, t_q) \end{array} & \to & \boxed{\begin{array}{c} L'_{t'_q} = [P'_{\geq 2}(L'^q), L'] \\ \Psi'_{t'_q} = P'_{\geq 2}(L'^q)\Psi' \end{array}} \\
\downarrow & & & & \downarrow \\
\begin{array}{c} \hat{L} = \Phi_x^{-1}\partial_x L \partial_x^{-1}\Phi_x \\ \hat{\Psi} = \Phi_x^{-1}\Psi_x \end{array} & & & & \begin{array}{c} \hat{L}' = \partial_x L \partial_x^{-1} \\ \hat{x}' = \Psi'_{x'} \ , \ \hat{t}'_q = t'_q \end{array} \\
\downarrow & & & & \downarrow \\
\boxed{\begin{array}{c} \hat{L}_{t_q} = [P_{\geq 1}(\hat{L}^q), \hat{L}] \\ \hat{\Psi}_{t_q} = P_{\geq 1}(\hat{L}^q)\hat{\Psi} \end{array}} & \to & \begin{array}{c} \hat{x}' = \hat{\Psi} \\ \hat{t}'_q = t_q, \ \hat{L}' = \hat{L} \end{array} & \to & \boxed{\hat{L}'_{\hat{t}'_q} = [\hat{P}'_{\geq 2}(\hat{L}'^q), \hat{L}']}
\end{array}
$$

Fig 2. The Reciprocal/Darboux Diagram
The mKP and Dym Hierarchies

In the above, a, b, c and d are constants introduced in Remarks 1 and 2. The projections $P'_{\geq 2}$, $\tilde{P}'_{\geq 2}$ and $\hat{P}'_{\geq 2}$ refer to powers of the differential symbols related to the variables x', $\tilde{x}'$ and $\hat{x}'$.

We finally emphasize the connection between the gauge transformations given by Theorems 3c) and 5d). These particular transformations are distinguished from the other results of these theorems by the commutativity property discussed in Section 5. Starting with an operator L and two eigenfunction Φ, Ψ associated with the modified KP hierarchy, we use Theorem 3c) with Φ to generate a further solution $\hat{L}$ of the modified KP hierarchy. Now Theorem 4 is used with the eigenfunction Ψ to generate a solution L' of the Dym hierarchy. The linked eigenfunction Φ' then is used via Theorem 5d) to generate a second Dym solution. It is readily checked that the resulting diagram commutes.

$$
\begin{array}{ccccc}
\underline{k=1} & & & & \underline{k=2} \\[1em]
\boxed{\begin{array}{c} L_{t_q} = [P_{\geq 1}(L^q), L] \\ \Phi_{t_q} = P_{\geq 1}(L^q)\Phi \\ \Psi_{t_q} = P_{\geq 1}(L^q)\Psi \end{array}} & \rightarrow & \begin{array}{c} x' = \Psi,\ t'_q = t_q,\ L' = L \\ \Phi'(x', t'_q) = \Phi(x, t_q) \end{array} & \rightarrow & \boxed{\begin{array}{c} L'_{t'_q} = [P'_{\geq 2}(L'^q), L'] \\ \Phi'_{t'_q} = P'_{\geq 2}(L'^q)\Phi' \end{array}} \\
\downarrow & & & & \downarrow \\
\begin{array}{c} \hat{L} = \frac{\Phi^2}{\Phi_x}\partial_x \frac{1}{\Phi} L \Phi \partial_x^{-1} \frac{\Phi_x}{\Phi^2} \\ \hat{\Psi} = \Psi - \frac{\Psi_x \Phi}{\Phi_x} \end{array} & & & & \begin{array}{c} \hat{L}' = \frac{\Phi'^2}{\Phi'_{x'}}\partial_{x'} \frac{1}{\Phi'} L' \Phi' \partial_{x'}^{-1} \frac{\Phi'_{x'}}{\Phi^2} \\ \hat{x}' = x' - \frac{\Phi'}{\Phi'_{x'}}\ ,\ \hat{t}'_q = t'_q \end{array} \\
\downarrow & & & & \downarrow \\
\boxed{\begin{array}{c} \hat{L}_{t_q} = [P_{\geq 1}(\hat{L}^q), \hat{L}] \\ \hat{\Psi}_{t_q} = P_{\geq 1}(\hat{L}^q)\hat{\Psi} \end{array}} & \rightarrow & \begin{array}{c} \hat{x}' = \hat{\Psi} \\ \hat{t}'_q = t_q,\ \hat{L}' = \hat{L} \end{array} & \rightarrow & \boxed{\hat{L}'_{\hat{t}'_q} = [\hat{P}'_{\geq 2}(\hat{L}'^q), \hat{L}']}
\end{array}
$$

Fig 3.
A Commutative mKP-Dym Hierarchy Link

Having obtained the Miura and auto-Bäcklund transformations as well as reciprocal links involving a change of the independent variables in terms of gauge-transformed Lax operators, it is next necessary to obtain these results explicitly for the fields satisfying the integrable equations. This now is a straightforward calculation by parametrizing the Lax operators as indicated in Section 2.

4 Results for 2+1-dimensional integrable hierarchies

4.1 The auto-Bäcklund transformation for the KP hierarchy

We consider the operator

$$L = L_{KP} = \partial_x + \frac{1}{2}u\partial_x^{-1} + u_2\partial_x^{-2} + u_3\partial_x^{-3} + \ldots \quad , \tag{4.1}$$

so that $L_{t_q} = [P_{\geq 0}(L^q), L]$ implies the KP equation (2.12) for the field u as indicated in Section 2.1. Denoting $t_1 = x, t_2 = y, t_3 = t$, we consider an eigenfunction $\Phi(x, y, t)$ satisfying $\Phi_{t_q} = P_{\geq 0}(L^q)\Phi$, that is

$$\Phi_y = \Phi_{xx} + u\Phi \quad , \quad \Phi_t = \Phi_{xxx} + \frac{3}{2}u\Phi_x + \frac{3}{2}u_x\Phi + 3u_2\Phi \tag{4.2}$$

(using (2.8)). Here u_2 is the first auxiliary field in (4.1), to be expressed by u via (2.10), that is $u_y = u_{xx} + 4u_{2x}$. According to Theorem 2 the new operator $\tilde{L} = \Phi\partial_x\Phi^{-1}L\Phi\partial_x^{-1}\Phi^{-1}$ will again satisfy the KP hierarchy. Calculating the leading order terms of $\tilde{L}$ we find

$$\tilde{L} = \partial_x \ + \ \frac{1}{2}(u + 2(ln(\Phi))_{xx})\ \partial_x^{-1} \ + \ (u_2 + \frac{1}{2}u_x - u\frac{\Phi_x}{\Phi} - \frac{\Phi_x\Phi_{xx}}{\Phi^2} + \frac{\Phi_x^3}{\Phi^3})\ \partial_x^{-2} \ + \ \ldots . \tag{4.3}$$

We conclude that the coefficients

$$\begin{array}{rcl} \tilde{u} & = & u + 2(ln(\Phi))_{xx} \ , \\ \tilde{u}_2 & = & u_2 + \frac{1}{2}u_x - u\Phi^{-1}\Phi_x - \Phi^{-2}\Phi_x\Phi_{xx} + \Phi^{-3}\Phi_x^3 \ , \\ \tilde{u}_3 & = & \ldots \ , \\ & \ldots & \end{array} \tag{4.4}$$

of $\tilde{L}$ will again satisfy the equations (2.10), (2.11). In particular, $\tilde{u}$ will again satisfy the KP (2.12). Hence,

$$\tilde{u} = u + 2(ln(\Phi))_{xx} \quad , \quad \Phi_y = \Phi_{xx} + u\Phi \tag{4.5}$$

represents the "spatial" part of the gauge transformation described by Theorem 2. Eliminating Φ in (4.5) one retrieves the auto-Bäcklund transformation of the KP. The "temporal" parts of the auto-Bäcklund transformations are obtained on use of $\Phi_{t_q} = P_{\geq 0}(L^q)\Phi$ for $q > 2$ instead of the y-evolution in (4.5). In this analysis, the auxiliary fields $u_2, u_3, ..$ occurring in the differential operators $P_{\geq 0}(L^q)$ have to be eliminated via (2.10).

4.2 The Miura link between KP and modified KP

We again consider the operator (4.1) and an eigenfunction Φ satisfying (4.2). According to Theorem 1 we now perform the gauge transformation $\tilde{L} = \Phi^{-1}L\Phi$, so that $\tilde{L}$ will satisfy the modified KP hierarchy. One calculates

$$\tilde{L} = \partial_x \ + \ \Phi^{-1}\Phi_x \ + \ \frac{1}{2}u\ \partial_x^{-1} \ + \ (u_2 - \frac{1}{2}u\ \Phi^{-1}\Phi_x)\ \partial_x^{-2} \ + \ (\ldots)\ \partial_x^{-3} \ + \ \ldots \tag{4.6}$$

as the leading orders of this operator. On comparison with the operator

$$L_{mKP} = \partial_x + v + v_1\partial_x^{-1} + v_2\partial_x^{-2} + v_3\partial_x^{-3} + \ldots \quad , \tag{4.7}$$

of Section 2.2 we conclude that the coefficients

$$\begin{array}{rcl} v & = & \Phi^{-1}\Phi_x \ , \\ v_1 & = & \frac{1}{2}u \ , \\ v_2 & = & u_2 - \frac{1}{2}u\ \Phi^{-1}\Phi_x \ , \\ & \ldots & \end{array} \tag{4.8}$$

of $\tilde{L}$ will satisfy the equations (2.15), (2.16). In particular, v will again satisfy the modified KP (2.17). Hence,

$$v = \Phi^{-1}\Phi_x \quad , \quad \Phi_y = \Phi_{xx} + u\Phi \tag{4.9}$$

represents the spatial part of the gauge transformation described by Theorem 1. Eliminating Φ in (4.9) one derives

$$(u + v_x + v^2)_x = v_y \quad , \tag{4.10}$$

that is the well known Miura-type transformation between the KP and modified KP equations ([39]). Again, the temporal parts of the Bäcklund transformations linking the KP and modified KP equations are obtained using $\Phi_{t_q} = P_{\geq 0}(L^q)\Phi$ with higher q's and eliminating the auxiliary fields by means of (2.10).

4.3 Auto-Bäcklund transformations for the modified KP hierarchy

Here, we start with the operator

$$L = L_{mKP} = \partial_x + v + v_1\partial_x^{-1} + v_2\partial_x^{-2} + v_3\partial_x^{-3} + \dots \quad , \tag{4.11}$$

so that $L_{t_q} = [P_{\geq 1}(L^q), L]$ implies the modified KP equation (2.17) for the field v as indicated in Section 2.2. Denoting $t_1 = x, t_2 = y, t_3 = t$, we consider an eigenfunction $\Phi(x,y,t)$ satisfying $\Phi_{t_q} = P_{\geq 1}(L^q)\Phi$, that is

$$\Phi_y = \Phi_{xx} + 2v\Phi_x \quad , \quad \Phi_t = \Phi_{xxx} + 3v\Phi_{xx} + 3(v_x + v^2 + v_1)\Phi_x \tag{4.12}$$

(using (2.13)). Here v_1 is the first auxiliary field in (4.11), to be expressed by v via (2.15), that is $v_y = v_{xx} + 2vv_x + 2v_{1x}$.

Theorem 3 describes three different gauge transformations leaving the modified KP hierarchy invariant:

Case a): According to Theorem 3a) the new operator $\tilde{L} = \Phi^{-1}L\Phi$ will again satisfy the modified KP hierarchy. Calculating the leading order terms of $\tilde{L}$ we find

$$\tilde{L} = \partial_x \; + \; (v + \Phi^{-1}\Phi_x) \; + \; v_1\,\partial_x^{-1} \; + \; (v_2 - \Phi^{-1}\Phi_x)\,\partial_x^{-2} \; + \; \dots \, . \tag{4.13}$$

We conclude that the coefficients

$$\begin{array}{rcl} \tilde{v} & = & v + \Phi^{-1}\Phi_x \quad , \\ \tilde{v}_1 & = & v_1 \quad , \\ \tilde{v}_2 & = & v_2 - \Phi^{-1}\Phi_x \quad , \\ & \dots & \end{array} \tag{4.14}$$

of $\tilde{L}$ will again satisfy the equations (2.15), (2.16). In particular, $\tilde{v}$ will again satisfy the modified KP (2.17). Hence,

$$\tilde{v} = v + \Phi^{-1}\Phi_x \quad , \quad \Phi_y = \Phi_{xx} + 2v\Phi_x \tag{4.15}$$

represents the spatial part of the gauge transformation described by Theorem 3a). Eliminating Φ in (4.15) one derives the spatial part

$$(\; (\tilde{v} - v)_x + (\tilde{v} - v)(\tilde{v} + v) \;)_x = (\tilde{v} - v)_y \quad . \tag{4.16}$$

of the auto-Bäcklund transformation for the modified KP which stems from the gauge transformation of Theorem 3a).

Case b): According to Theorem 3b) the new operator $\tilde{L} = \Phi_x^{-1}\partial_x L \partial_x^{-1}\Phi_x$ will again satisfy the modified KP hierarchy. Calculating the leading order terms of $\tilde{L}$ we find

$$\tilde{L} = \partial_x \; + \; (v + \Phi_x^{-1}\Phi_{xx}) \; + \; (v_1 + v_x)\,\partial_x^{-1} \; + \; \dots \, . \tag{4.17}$$

We conclude that the coefficients

$$\begin{array}{rcl} \tilde{v} & = & v + \Phi_x^{-1}\Phi_{xx} \quad , \\ \tilde{v}_1 & = & v_1 + v_x \quad , \\ & \dots & \end{array} \tag{4.18}$$

of $\tilde{L}$ will again satisfy the equations (2.15), (2.16). In particular, $\tilde{v}$ will again satisfy the modified KP (2.17). Hence,

$$\tilde{v} = v + \Phi_x^{-1}\Phi_{xx} \quad , \quad \Phi_y = \Phi_{xx} + 2v\Phi_x \tag{4.19}$$

represents the spatial part of the gauge transformation. Eliminating Φ in (4.19) one derives the spatial part

$$(\ (\tilde{v}+v)_x + (\tilde{v}-v)(\tilde{v}+v)\ )_x = (\tilde{v}-v)_y \quad . \tag{4.20}$$

of the auto-Bäcklund transformation for the modified KP given by the gauge transformation of Theorem 3b).

Case c): According to Theorem 3c) the new operator $\tilde{L} = \Phi_x^{-1}\Phi^2\partial_x\Phi^{-1}L\Phi\partial_x^{-1}\Phi_x\Phi^{-2}$ will again satisfy the modified KP hierarchy. Calculating the leading order terms of $\tilde{L}$ we find

$$\tilde{L} = \partial_x \ + \ (v + \Phi_x^{-1}\Phi_{xx} - \Phi^{-1}\Phi_x) \ + \ (v_1 + v_x + (\Phi^{-1}\Phi_x)_x)\ \partial_x^{-1} \ + \ \ldots . \tag{4.21}$$

We conclude that the coefficients

$$\begin{array}{rcl} \tilde{v} &=& v + \Phi_x^{-1}\Phi_{xx} - \Phi^{-1}\Phi_x \quad , \\ \tilde{v}_1 &=& v_1 + v_x + (\Phi^{-1}\Phi_x)_x \quad , \\ & \ldots & \end{array} \tag{4.22}$$

of $\tilde{L}$ will again satisfy the equations (2.15), (2.16). In particular, $\tilde{v}$ will again satisfy the modified KP (2.17). Hence,

$$\tilde{v} = v + \Phi_x^{-1}\Phi_{xx} - \Phi^{-1}\Phi_x \quad , \quad \Phi_y = \Phi_{xx} + 2v\Phi_x \tag{4.23}$$

represents the spatial part of the gauge transformation described by Theorem 3c). In order to eliminate Φ in (4.23) one may first rewrite (4.23) as

$$\tilde{v} - v = \Phi_x^{-1}\Phi_{xx} - \Phi^{-1}\Phi_x \quad , \quad \tilde{v} + v = \Phi_x^{-1}\Phi_y - \Phi^{-1}\Phi_x \quad . \tag{4.24}$$

At this stage it is interesting to investigate the 1+1-dimensional limit of these equations. We note that the evolution equation $\Phi_y = \Phi_{xx} + 2v\Phi_x$ for the eigenfunction Φ turns into a proper eigenfunction equation by imposing the constraint $\Phi_y = \lambda^2\Phi$ with spectral parameter λ. This corresponds to the "2-reduction" $L^2 = P_{\geq 1}(L^2)$, which implies direct relations between the auxiliary fields $v_1, v_2, \ldots$ and the prime field v. For example, one finds $v_x + v^2 + 2v_1 = 0$ as the 0-order of L^2, so that $v_1 = -(v_x + v^2)/2$. In this reduction the operator $L_{mKdV} = L^2 = \partial_x^2 + 2v\partial_x$ is the Lax operator for the modified Korteweg-de Vries equation. Indeed, with

$$P_{\geq 1}(L_{mKdV}^{3/2}) = P_{\geq 1}(L^3) = \partial_x^3 + 3v\partial_x^2 + \frac{3}{2}(v_x + v^2)\partial_x \tag{4.25}$$

one finds

$$(L_{mKdV})_t = [P_{\geq 1}(L_{mKdV}^{3/2}), L_{mKdV}] \quad \leftrightarrow \quad 4v_t = v_{xxx} - 6v^2 v_x \quad , \tag{4.26}$$

which may be understood as the reduction $v_y = 0$ of the modified KP equation (2.17). In this case the gauge transformation (4.24) with $\Phi_y = \lambda^2\Phi$ may be simplified by introducing

$$a = \lambda^{-1}\ \Phi^{-1}\Phi x \quad . \tag{4.27}$$

Now (4.24) reads

$$\tilde{v} - v = a^{-1}a_x \quad , \quad \tilde{v} + v = \lambda(a^{-1} - a) \quad , \tag{4.28}$$

whence $a = exp(\int^x(\tilde{v} - v))$ and the auto-Bäcklund transformation assumes the form

$$\tilde{v} + v = -2\,\lambda\ sinh(\int^x(\tilde{v} - v)) \quad . \tag{4.29}$$

On the substitution $(v,\tilde{v}) \to (\sqrt{-1}\,v, \sqrt{-1}\,\tilde{v})$, so that (4.26) is changed into $4v_t = v_{xxx} + 6v^2 v_x$, we retrieve the spatial part of the auto-Bäcklund transformation shared with the sine-Gordon equation ([40]).

Thus, returning to (4.24) without any reduction assumptions, this transformation represents the 2+1-dimensional generalization of (4.29). To eliminate Φ we may again introduce $a = \Phi^{-1}\Phi_x$, turning (4.24) into

$$\tilde{v} - v = a^{-1} a_x \quad , \quad \tilde{v} + v = a^{-1} \int^x a_y \; - \; a \quad . \tag{4.30}$$

The first equation may be integrated to eliminate a from this transformation. As shall be discussed in Section 5, the transformation (4.30) is distinguished from (4.16) and (4.20) by the fact that only for (4.30) will the Bianchi diagram commute.

4.4 The link between modified KP and Dym hierarchies

We again start with the operator $L(x,y,t)$ given by (4.11) and an eigenfunction $\Phi(x,y,t)$ satisfying (4.12). According to Theorem 4 we may introduce the transformation

$$x' = \Phi(x,y,t) \quad , \quad y' = y \quad , \quad t' = t \quad , \tag{4.31}$$

so that the operator $L'(x',y',t') = L(x,y,t)$ will describe the Dym hierarchy. Using $\partial_x = \Phi_x \partial_{x'}$ one calculates the leading order terms of L' as

$$L' = \Phi_x \partial_{x'} \; + \; v \; + \; v_1 \Phi_x^{-1} \, \partial_{x'}^{-1} \; + \; \ldots \, . \tag{4.32}$$

Hence we conclude that the coefficients

$$\begin{array}{rcl} r & = & \Phi_x \;\; , \\ r_0 & = & v \;\; , \\ r_1 & = & v_1 \Phi_x^{-1} \quad , \\ & \cdots & \end{array} \tag{4.33}$$

of L' satisfy the equations (2.20), (2.21). In particular, r will satisfy the Dym equation (2.22). Hence,

$$x' = \Phi \quad , \quad y' = y \quad , \quad t' = t \quad , \quad r = \Phi_x \quad , \quad \Phi_y = \Phi_{xx} + 2v\Phi_x \tag{4.34}$$

represents the "spatial" part of the reciprocal transformation described by Theorem 4. Eliminating Φ in (4.34) one finds

$$(2vr + r_x)_x = r_y \quad , \quad \frac{\partial x'}{\partial x} = r \tag{4.35}$$

as the equations linking a solution v of the modified KP to a solution r of the Dym equation (2.22) via the change of variables $x \to x'$. Assuming the eigenfunction $\Phi(t_q)$, $q \in \mathbb{N}$, to depend on all evolution parameters of the modified KP hierarchy via $\Phi_{t_q} = P_{\geq 1}(L^q)\Phi$, we may express this change of variables infinitesimally in the following compact form:

$$dx' = \sum_q \Phi_{t_q} dt_q = \sum_q \; [\![P_{\geq 1}(L^q)\Phi]\!] \; dt_q = r dx + (r_x + 2vr) dy + \sum_{q \geq 3} [\![P_{\geq 1}(L^q)\partial_x^{-1} r]\!] \; dt_q \; . \tag{4.36}$$

Here all auxiliary variables $v_1, v_2, \ldots$ occurring in $P_{\geq 1}(L^q)$ can be expressed in terms of v by (2.15). A reciprocal link between the KP and the Dym equation was first observed in [25]. There the independent variable x' of the Dym equation is obtained as $x' = \Phi$, where Φ is the singularity field arising from the Painlevé analysis of the **KP** equation. Thus, it might seem surprising that in Theorem 4 we have $x' = \Phi$, where Φ is an eigenfunction for the modified KP. The explanation is simple: if we consider an eigenfunction Φ satisfying (4.12), we may use the first linear equation to express v as $2v = \Phi_x^{-1}(\Phi_y - \Phi_{xx})$. Inserting this into (2.15), that is $2v_{1x} = v_y - v_{xx} - 2vv_x$, we can eliminate both v and v_1 from the time evolution Φ_t in (4.12). The resulting equation is

$$4\left(\frac{\Phi_t}{\Phi_x}\right)_x = \left(\frac{\Phi_{xxx}}{\Phi_x} - \frac{3}{2}\left(\frac{\Phi_{xx}}{\Phi_x}\right)^2 + \frac{3}{2}\left(\frac{\Phi_y}{\Phi_x}\right)^2\right)_x + 3\left(\frac{\Phi_y}{\Phi_x}\right)_y \quad , \tag{4.37}$$

which is precisely the singularity manifold equation for the KP as found by Weiss [41]. Thus, the singularity field Φ of the KP can be regarded as an eigenfunction associated with a solution $v = \frac{1}{2}\Phi_x^{-1}(\Phi_y - \Phi_{xx})$ of the modified KP.

4.5 Auto-Bäcklund transformations for the Dym hierarchy

Now we start with the operator $L(x,y,t)$ given by

$$L = L_D = r\partial_x + r_0 + r_1\partial_x^{-1} + r_2\partial_x^{-2} + r_3\partial_x^{-3} + \ldots \quad , \tag{4.38}$$

representing the Dym equation (2.22) via $L_{t_q} = [P_{\geq 2}(L^q), L]$, $y = t_2, t = t_3$. We consider an eigenfunction $\Phi(x,y,t)$ satisfying $\Phi_{t_q} = P_{\geq 2}(L^q)\Phi$, that is

$$\Phi_y = r^2\Phi_{xx} \quad , \quad \Phi_t = r^3\Phi_{xxx} + 3r^2(r_x + r_0)\Phi_{xx} \tag{4.39}$$

(using (2.18)). Here r_0 is the first auxiliary field in (4.38), to be expressed by r via (2.20), that is $r_y = r^2 r_{xx} + 2r^2 r_{0x}$.

Theorem 5 describes 4 different gauge transformations leaving the Dym hierarchy invariant. We summarize the cases a) and b) of Theorem 5 according to Remark 2 and discuss the following three cases:

Cases a) and b) : According to Remark 2 the transformation

$$x \;\to\; x' = (cx+d)^{-1}(a\Phi + b) \quad , \quad L \;\to\; L' = (cx+d)^{-1}L(cx+d) \tag{4.40}$$

with arbitrary constants a, b, c and d will leave the Dym hierarchy invariant. Calculating the leading order terms of L' we find

$$L' = r\frac{\partial x'}{\partial x}\partial_{x'} \;+\; (r_0 + \frac{cr}{cx+d}) \;+\; r_1(\frac{\partial x'}{\partial x})^{-1}\,\partial_{x'}^{-1} \;+\; \ldots\,. \tag{4.41}$$

We conclude that the coefficients

$$\begin{array}{rcl} r' & = & r\frac{\partial x'}{\partial x} \quad , \\ r_0' & = & r_0 + \frac{cr}{cx+d} \quad , \\ r_1' & = & r_1(\frac{\partial x'}{\partial x})^{-1} \quad , \\ & \ldots & \end{array} \tag{4.42}$$

of L' will again satisfy the equations (2.20), (2.21). In particular, r' will again satisfy the Dym equation (2.22). Hence,

$$x' = (cx+d)^{-1}(a\Phi + b) \quad , \quad r' = r\;\frac{acx\Phi_x + ad\Phi_x - ac\Phi - cb}{(cx+d)^2} \quad , \quad \Phi_y = r^2\Phi_{xx} \tag{4.43}$$

represents the spatial part of the reciprocal transformation described by Theorem 5a)/b). In particular, specializing the constants a, b, c, d and the eigenfunction Φ one finds the following transformations leaving the Dym equation (2.22) invariant:

$$
\begin{array}{llll}
\text{i)} & a=1,\ b=0,\ c=0,\ d=1\ : & x'=\Phi & ,\quad r'=r\,\Phi_x\ , \\
\text{ii)} & a=1,\ b=0,\ c=1,\ d=0\ : & x'=\frac{\Phi}{x} & ,\quad r'=r\,(\frac{\Phi}{x})_x\ , \\
\text{iii)} & \Phi=x\ : & x'=\frac{ax+b}{cx+d} & ,\quad r'=r\,\frac{ad-bc}{(cx+d)^2}\ ,
\end{array}
$$

where Φ has to satisfy (4.39). Case iii) is the invariance triggered by a Möbius transformation of the space variable as observed in [25]. We remark that according to (4.40) this transformation is just a special case of a much larger class of transformations involving an eigenfunction Φ. For the cases i) and ii) (and the more general (4.40)) one may characterize these transformations directly by eliminating Φ via (4.39). For example, for case i) one has $\Phi_x = r^{-1}r'$, inserting this into $\Phi_y = r^2\Phi_{xx}$ one obtains

$$
i)\qquad \frac{\partial x}{\partial x'} = \frac{r'}{r}\ ,\quad \left(\frac{r'}{r}\right)_y = (r'_x r - r' r_x)_x\ . \tag{4.44}
$$

For case ii) we find $\Phi_{xx} = 2r^{-1}r' + x(r^{-1}r')_x$, hence

$$
ii)\qquad \frac{\partial x}{\partial x'} = \frac{r'}{r}\ ,\quad \left(\frac{r'}{r}\right)_y = (2\frac{rr'}{x} + r'_x r - r' r_x)_x\ . \tag{4.45}
$$

Again, assuming the eigenfunction $\Phi(t_q)$, $q \in \mathbb{N}$, to depend on all evolution parameters of the Dym hierarchy via $\Phi_{t_q} = P_{\geq 2}(L^q)\Phi$, we may express these changes of variables infinitesimally in the following compact forms:

$$
\begin{array}{lll}
i) & dx' & = \Phi_x dx\ +\ \sum_{q\geq 2}\Phi_{t_q}dt_q = \Phi_x dx\ +\ \sum_{q\geq 2}\ [\![P_{\geq 2}(L^q)\Phi]\!]\ dt_q \\
 & & = \frac{r'}{r}dx\ +\ (r'_x r - r' r_x)\ dy\ +\ \sum_{q\geq 3}[\![P_{\geq 2}(L^q)\partial_x^{-1}\frac{r'}{r}]\!]\ dt_q\ , \\
ii) & dx' & = (\frac{\Phi}{x})_x dx\ +\ \sum_{q\geq 2}(\frac{\Phi}{x})_{t_q}dt_q = (\frac{\Phi}{x})_x dx\ +\ \sum_{q\geq 2}\ \frac{1}{x}[\![P_{\geq 2}(L^q)\Phi]\!]\ dt_q \\
 & & = \frac{r'}{r}dx\ +\ (2\frac{rr'}{x} + r'_x r - r' r_x)\ dy\ +\ \sum_{q\geq 3}\frac{1}{x}[\![P_{\geq 2}(L^q)(\partial_x^{-1}\ x\ \frac{r'}{r} + \partial_x^{-2}\frac{r'}{r})]\!]\ dt_q\ .
\end{array} \tag{4.46}
$$

Here all auxiliary variables $r_0, r_1, \ldots$ occurring in $P_{\geq 2}(L^q)$ can be expressed in terms of r by (2.20).

Case c): According to Theorem 5c) the transformation

$$
x\ \to\ x' = \Phi_x\ \ ,\ \ L\ \to\ L' = \partial_x L \partial_x^{-1} \tag{4.47}
$$

will leave the Dym hierarchy invariant. Calculating the leading order terms of L' we find

$$
L' = r\Phi_{xx}\ \partial_x\ +\ (r_0 + r_x)\ +\ (r_1 + r_{0x})\ \Phi_{xx}^{-1}\ \partial_x^{-1}\ +\ \ldots\ . \tag{4.48}
$$

We conclude that the coefficients

$$
\begin{array}{rcl}
r' & = & r\Phi_{xx}\ , \\
r'_0 & = & r_0 + r_x\ , \\
r'_1 & = & (r_1 + r_{0x})\ \Phi_{xx}^{-1}\ , \\
 & \cdots &
\end{array} \tag{4.49}
$$

of L' will again satisfy the equations (2.20), (2.21). In particular, r' will again satisfy the Dym equation (2.22). Hence,

$$
x' = \Phi_x\ \ ,\ \ r' = r\Phi_{xx}\ \ ,\ \ \Phi_y = r^2\Phi_{xx} \tag{4.50}
$$

represents the spatial part of the reciprocal transformation described by Theorem 5c). Eliminating Φ in (4.50) one derives the spatial part

$$\frac{\partial x'}{\partial x} = \frac{r'}{r} \quad , \quad \left(\frac{r'}{r}\right)_y = (rr')_{xx} \tag{4.51}$$

of this transformation.
Assuming the eigenfunction $\Phi(t_q)$, $q \in \mathbb{N}$, to depend on all evolution parameters of the Dym hierarchy via $\Phi_{t_q} = P_{\geq 2}(L^q)\Phi$, we may express this change of variables infinitesimally in the following compact form:

$$\begin{aligned} dx' &= \Phi_{xx}dx + \textstyle\sum_{q\geq 2} \Phi_{xt_q}dt_q = \Phi_{xx}dx + \sum_{q\geq 2} \ [\![P_{\geq 2}(L^q)\Phi]\!]_x \ dt_q \\ &= \tfrac{r'}{r}dx + (rr')_x \ dy + \textstyle\sum_{q\geq 3} [\![P_{\geq 2}(L^q)\partial_x^{-2}\tfrac{r'}{r}]\!]_x \ dt_q \ , \end{aligned} \tag{4.52}$$

Here all auxiliary variables $r_0, r_1, ...$ occurring in $P_{\geq 2}(L^q)$ can be expressed in terms of r by (2.20).

We remark that, in a fashion similar to (4.25)-(4.28), we may impose "2-reduction" on the 2+1-dimensional Dym hierarchy, that is we impose the restriction $L^2 = P_{\geq 2}(L^2)$ on the Lax operator (4.38). This leaves just the prime field r as an unconstrained field, all auxiliary fields $r_0, r_1, ..$ in (4.38) are related to r by differential relations. For example, calculating

$$L^2 = r^2\partial_x^2 \ + \ (rr_x + 2rr_0)\ \partial_x \ + \ (rr_{0x} + r_0^2 + 2rr_1) \ + \ (..)\ \partial_x^{-1} \ + \ ... \tag{4.53}$$

we have to assume $r_0 = -\frac{1}{2}r_x$ and $rr_1 = \frac{1}{4}rr_{xx} - \frac{1}{8}r_x^2$ to satisfy $L^2 = r^2\partial_x^2$. Equations (2.20) now reduce to $r_y = 0, r_{0y} = 0, ..$ and (2.22) describes the 1+1-dimensional Dym equation

$$4r_t = r^3 r_{xxx} \quad . \tag{4.54}$$

Assuming $\Phi_y = \lambda\Phi$ the evolutions (4.39) turn into the Lax pair

$$r^2\Phi_{xx} = \lambda\Phi \quad , \quad \Phi_t = r^3\Phi_{xxx} + \frac{3}{2}r^2 r_x \Phi_{xx} \tag{4.55}$$

for (4.54). Since constants and x will satisfy the time evolutions $\Phi_{t_q} = P_{\geq 2}(L^q)\Phi$ for eigenfunctions trivially, we may replace an eigenfunction Φ by $\Phi + const_1 + const_2\ x$, leading to the following Lax pair

$$r^2\Phi_{xx} = \lambda_0 + \lambda_1 x + \lambda_2\Phi \quad , \quad \Phi_t = r^3\Phi_{xxx} + \frac{3}{2}r^2 r_x \Phi_{xx} \tag{4.56}$$

for (4.54) involving three spectral parameters λ_0, λ_1 and λ_2. Now the transformation (4.50) may be rewritten as

$$\frac{\partial x'}{\partial x} = \frac{r'}{r} \quad , \quad rr' = \lambda_0 + \lambda_1 x + \lambda_2\Phi \quad . \tag{4.57}$$

For $\lambda_0 = 1$, $\lambda_1 = \lambda_2 = 0$ we retrieve the reciprocal invariance of the Dym equation

$$r' = \frac{1}{r} \quad , \quad x' = \int^x \frac{1}{r^2} \tag{4.58}$$

originally derived in [22].

Case d): According to Theorem 5d) the transformation

$$x \ \to \ x' = x - \Phi_x^{-1}\Phi \quad , \quad L \ \to \ L' = \Phi_x^{-1}\Phi^2\partial_x\Phi^{-1}L\Phi\partial_x^{-1}\Phi_x\Phi^{-2} \tag{4.59}$$

will leave the Dym hierarchy invariant. Calculating the leading order terms of L' we find

$$L' = r\,\frac{\Phi\Phi_{xx}}{\Phi_x^2}\,\partial_x \;+\; (r_0 + r_x - r\frac{\Phi_x}{\Phi} + r\frac{\Phi_{xx}}{\Phi_x}) \;+\; (..)\,\partial_x^{-1} \;+\; \dots \,. \tag{4.60}$$

We conclude that the coefficients

$$\begin{aligned} r' &= r\,\Phi_x^{-2}\Phi\Phi_{xx} \;, \\ r_0' &= r_0 + r_x - r\,\Phi^{-1}\Phi_x + r\,\Phi_x^{-1}\Phi_{xx} \;, \\ &\dots \end{aligned} \tag{4.61}$$

of L' will again satisfy the equations (2.20), (2.21). In particular, r' will again satisfy the Dym equation (2.22). Hence,

$$x' = x - \Phi_x^{-1}\Phi \;\;,\;\; r' = r\,\Phi_x^{-2}\Phi\Phi_{xx} \;\;,\;\; \Phi_y = r^2\Phi_{xx} \tag{4.62}$$

represents the spatial part of the reciprocal transformation described by Theorem 5d). Although this transformation looks somewhat more involved than the cases a)-c) above, (4.62) is distinguished from the other cases by the fact that the Bianchi diagram associated with (4.62) will commute. This will be discussed in the next section.

5 Remarks on the commutativity of the Bianchi diagrams

The gauge transformations and reciprocal links of Section 4 do not just provide Miura- and auto-Bäcklund transformations. In fact, Theorems 1-5 represent Darboux theorems for the integrable hierarchies under consideration, that is, known eigenfunctions of an operator L of (2.5) not only lead to a new operator $\tilde{L}$, but also to new eigenfunctions for $\tilde{L}$. Hence the process of generating new operators can be continued, if a sufficiently large number of eigenfunctions is known for the original L. In particular, starting with two eigenfunctions Φ and Ψ, one may use Φ to obtain a gauge-transformed $\tilde{L}$ with eigenfunction $\tilde{\Psi}$, and then use $\tilde{\Psi}$ to obtain a further operator L', say. The roles of the two functions Φ and Ψ can be swapped and it is natural to inquire, as to or not the two results for L' will coincide. Thus, it is known that commutativity of Bianchi diagrams leads to nonlinear superposition principles whereby exact solutions can be generated by purely algebraic procedures.

The transformations of Theorems 2,3 and 5 are all of the form

$$\begin{aligned} \mathbf{B}(\Phi):\;\; L \;&\rightarrow\; \tilde{L} \;=\; G(\Phi)LG(\Phi)^{-1} \qquad , \quad \Psi \;\rightarrow\; \tilde{\Psi} = G(\Phi)\Psi \;, \\ \mathbf{B}(\tilde{\Psi}):\;\; \tilde{L} \;&\rightarrow\; L' \;=\; G(\tilde{\Psi})\tilde{L}G(\tilde{\Psi})^{-1} \;, \end{aligned} \tag{5.63}$$

where Φ and Ψ are eigenfunctions of L and $G(\Phi)$ is some gauge operator depending on Φ. Thus, for Theorem 2 we have $G(\Phi) = \Phi\partial_x\Phi^{-1}$, while for Theorems 3c) and 5d) we have $G(\Phi) = \Phi_x^{-1}\Phi^2\partial_x\Phi^{-1}$. We consider the following diagram

$$
\begin{tikzcd}
& \boxed{L, \Phi, \Psi} \arrow[dl, "\Phi"'] \arrow[dr, "\Psi"] & \\
\boxed{\begin{array}{c} \tilde{L} = \mathbb{B}(\Phi) L \\ \tilde{\Psi} = \mathbb{B}(\Phi)\Psi \end{array}} \arrow[dr, "\tilde{\Psi}"'] & & \boxed{\begin{array}{c} \hat{L} = \mathbb{B}(\Psi) L \\ \hat{\Phi} = \mathbb{B}(\Psi)\Phi \end{array}} \arrow[dl, "\hat{\Phi}"] \\
& \boxed{\mathbb{B}(\tilde{\Psi})\tilde{L} \;\; (\stackrel{?}{=}) \;\; \mathbb{B}(\hat{\Phi})\hat{L}} &
\end{tikzcd}
$$

Fig 4. The Bianchi Diagram

The question to be answered is whether this diagram will commute, that is $\mathbb{B}(\mathbb{B}(\Phi)\Psi))\mathbb{B}(\Phi) = \mathbb{B}(\mathbb{B}(\Psi)\Phi))\mathbb{B}(\Psi)$ for the transformations given by Theorems 2,3 and 5.

5.1 Commutativity for the auto-Bäcklund tranformation for the KP

The well-known commutativity of the Bianchi Diagram for the auto-Bäcklund transformation (4.5) for the KP hierarchy follows directly from Theorem 2. With $G(\Phi) = \Phi \partial_x \Phi^{-1}$ we have

$$L' = \mathbb{B}(\tilde{\Psi})\tilde{L} = G(\tilde{\Psi}) G(\Phi) L G(\Phi)^{-1} G(\tilde{\Psi})^{-1} \quad . \tag{5.64}$$

With $\tilde{\Psi} = \Phi(\Phi^{-1}\Psi)_x$ it is a simple calculation to check that

$$G(\tilde{\Psi}) G(\Phi) = \partial_x^2 \; - \; \frac{W(\Phi,\Psi)_x}{W(\Phi,\Psi)} \partial_x \; + \; \frac{W(\Phi_x,\Psi_x)}{W(\Phi,\Psi)} \quad , \tag{5.65}$$

where $W(f,g) = f g_x - f_x g$ is the Wronskian determinant of two functions. Thus, the final gauge operator linking L and L' is clearly symmetric in Φ and Ψ.

5.2 Results for the modified KP

We consider the transformation

$$\mathbb{B}(\Phi) : \qquad L \;\rightarrow\; \tilde{L} = \frac{\Phi^2}{\Phi_x} \partial_x \frac{1}{\Phi} L \Phi \partial_x^{-1} \frac{\Phi_x}{\Phi^2} \quad , \quad \Psi \;\rightarrow\; \tilde{\Psi} = \frac{\Psi_x \Phi - \Phi_x \Psi}{\Phi_x} \tag{5.66}$$

given in Theorem 3c), that is $G(\Phi) = \Phi_x^{-1} \Phi^2 \partial_x \Phi^{-1} = \Phi_x^{-1} \Phi \partial_x \; - \; 1$. By a simple calculation one checks that

$$G(\tilde{\Psi}) G(\Phi) = \frac{W(\Phi,\Psi)}{W(\Phi_x,\Psi_x)} \partial_x^2 \; - \; \frac{W(\Phi,\Psi)_x}{W(\Phi_x,\Psi_x)} \partial_x \; + \; 1 \quad , \tag{5.67}$$

which is again symmetric in Φ and Ψ, so that Diagram 4 commutes for (5.66). It is readily checked that such commutativity does not hold for the gauge transformations a) and b) of Theorem 3.

5.3 Results for the Dym hierarchy

We again consider the transformation (5.66), now supplemented by the change of variables $\tilde{x} = x - \Phi_x^{-1}\Phi$. This is the invariance of the Dym hierarchy given by Theorem 5d), the gauge operator is again $G(\Phi) = \Phi_x^{-1}\Phi\partial_x$. After the change of variable $x \to \tilde{x}$ we have to perform the second transformation with $G(\tilde{\Psi})$. We observe that

$$G(\tilde{\Psi}) \;=\; \frac{\tilde{\Psi}^2}{\tilde{\Psi}_{\tilde{x}}}\partial_{\tilde{x}}\frac{1}{\tilde{\Psi}} \;=\; \frac{\tilde{\Psi}^2}{\tilde{\Psi}_x}\partial_x\frac{1}{\tilde{\Psi}} \quad , \tag{5.68}$$

hence the final transformation $L \to L' = \mathbb{B}(\tilde{\Psi})\mathbb{B}(\Phi)L$ via $G(\tilde{\Psi})G(\Phi)$ is exactly the same as for the modified KP. From (5.67) we derive the symmetry of this transformation with respect to Φ and Ψ. It remains to be shown that the final space variable $x' = \tilde{x} - \tilde{\Psi}_{\tilde{x}}^{-1}\tilde{\Psi}$ associated with the operator L' is obtained in a symmetric way. A simple calculation yields

$$x' = x \;-\; \frac{W(\Phi,\Psi)_x}{W(\Phi_x,\Psi_x)} \quad . \tag{5.69}$$

Hence, commutativity of Diagram 4 obtains for the transformation given by Theorem 5d). We note that such a result does not hold for the other cases of Theorem 5. For example, if we consider Theorem 5c), we introduce $\tilde{x} = \Phi_x$ and $\tilde{\Psi} = \Psi_x$. Repeating this transformation with the eigenfunction $\tilde{\Psi}$, one finds

$$x' = \tilde{\Psi}_{\tilde{x}} = \frac{\tilde{\Psi}_x}{\frac{\partial \tilde{x}}{\partial x}} = \frac{\Psi_{xx}}{\Phi_{xx}} \quad . \tag{5.70}$$

This is clearly not symmetric in Φ and Ψ, but an additional transformation given by Theorem 5b) has to be included, when Φ and Ψ are exchanged.

6 Perspectives

In this paper, we have extended the construction of nonlinear integrable differential equations via Sato theory to three natural classes of Lax equations (2.5). The case $k = 0$ corresponds to the known construction of the KP hierarchy, here supplemented by the case $k = 1$ leading to the modified KP hierarchy. These two hierarchies are shown to be closely connected via gauge transformations involving eigenfunctions satisfying the associated linear equations. The third class $k = 2$ may be understood as originating from the modified KP hierarchy by a change of the independent space variable. Again eigenfunctions are involved in a natural way. Further, various gauge transformations and reciprocal links are found which leave these hierarchies invariant. They form a logical pattern of commuting Bäcklund diagrams emphasizing the intimate relations between the three hierarchies.

A set of Darboux Theorems is given which embody these links. For the KP case the Darboux Theorem is known to lead to a simple representation of exact solutions via Wronskian determinants of eigenfunctions. The additional results presented here will allow corresponding Wronskian formulations for the modified KP and the Dym hierarchy in an analogous way. In particular, classes of exact solutions for the Dym equation (2.22) can be derived in parametrized form. These results will be reported elsewhere.

The transformations discussed here are formulated in full generality for the Lax equations (2.5). An infinite set of coupled nonlinear equations involving infinitely many time-variables and infinitely many fields parametrizing the operators (2.7) is encoded in these Lax equations. One way of extracting closed equations is to consider one of the fields in L as a prime field, whereas all the remaining fields are regarded to be auxiliary fields which can be eliminated. This is the standard approach for extraction of the KP equation (2.12) out of the KP hierarchy (2.9)-(2.11). Here we have adopted this approach, showing that the modified KP equation (2.17) and the Dym equation (2.22) are simple realizations of more general hierarchies (2.5) in terms of 2+1-dimensional integrable equations.

Another way of extracting closed systems of nonlinear integrable models out of (2.5) consists of imposing additional constraints ("N-reduction") on the pseudo-differential operator. The simplest of these reductions lead to the Korteweg-de Vries and the Boussinesq equations as realizations of the general KP hierarchy. Corresponding reductions can be imposed on the general modified KP hierarchy and the Dym hierarchy, leading to a large variety of integrable 1+1-dimensional hierarchies. The general concepts of this paper apply to these reductions in a straightforward way. A general discussion of the reduction properties, corresponding gauge transformations and reciprocal links, as well as the Hamiltonian aspects of the Lax equations (2.5) can be found in [37].

The extension of the procedures of this paper to the 2+1-dimensional Caudrey-Dodd-Gibbon and Kaup-Kupershmidt hierarchies is under investigation. These systems are found when constraints of the form $\partial^k L = -L^\dagger \partial^k$, $k = 0, 1$ or 2, are imposed on the Lax operators (2.6) and (2.7). Here $\dagger$ is to denote transposition of the operators. However, the Darboux transformations considered here will not leave such constraints invariant. It seems that not only eigenfunctions but also adjoint eigenfunctions associated with the Lax operators have to be taken into account. They give rise to "binary" Darboux transformations (see [14] for the case of the KP equation) which may be used to generate invariances of Lax equations subject to such constraints. Details are to be published elsewhere. In 1+1-dimensions reciprocal links of such constrained systems have been established in [42].

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