in: *Applications of Analytic and Geometric Methods to Nonlinear Differential Equations* (Proceedings of the NATO Advanced Research Workshop, Exeter, 14-17 July 1992, UK), pp. 193-206, P.A. Clarkson (ed.), Kluwer Publ. (1993).

DARBOUX THEOREMS AND THE KP HIERARCHY

W. Oevel and W. Schief
Department of Mathematical Sciences
Loughborough University of Technology
Loughborough LE11 3TU
United Kingdom

ABSTRACT. Generalizations of the classical Darboux theorem are established for arbitrary ordinary matrix differential operators. Darboux transformations may be regarded as gauge transformations, where the gauge operator is a first order differential operator parametrized by an eigenfunction. Adjoint Darboux transformations triggered by adjoint eigenfunctions are introduced via transposition of the operators. The composition of a Darboux and an adjoint Darboux transformation leads to the notion of binary transformations, which are triggered by pairs of eigenfunctions and adjoint eigenfunctions. A formalism involving pseudo-differential symbols is used to give a general formulation of these transformations for arbitrary scattering operators. It is shown how exact solutions of the multicomponent KP hierarchy are generated from these transformations.

1. Introduction

Since the mid 1970's Darboux transformations (DTs) have been shown to be a powerful tool for studying the nonlinear integrable systems of soliton theory. Darboux's original result [1] is a transformation between Schrödinger equations with different potentials. Crum [2] considered the iteration of the DT, which can be formulated in a compact way involving the Wronskian determinant of eigenfunctions. The key fact leading to the spectacular properties of soliton systems is their connection to linear scattering problems. For the celebrated Korteweg-de Vries (KdV) equation the associated linear problem is the eigenvalue equation for the Schrödinger operator. Hence, it is not surprising that Darboux's result has implications for the KdV. In particular, it was shown by Crum that the DT adds an eigenvalue to the spectrum of the Schrödinger operator. On the level of the nonlinear equation this means that solitons are added by this transformation [3], and Crum's Wronskian formulas for the iterated DT correspond to compact Wronskian representations of the n-soliton solutions of the KdV [4].

Generalizations of these ideas for other spectral equations and their associated soliton systems have been successfully applied to a variety of systems. This includes spectral problems for ordinary differential operators, which are associated with soliton systems in 1+1-dimensions, as well as scattering problems in several variables and their associated nonlinear equations in higher dimensions. In particular, interesting localized coherent structures (dromions) were constructed using an appropriate Darboux formalism [5]. The recent book [6] by Matveev and Salle provides an excellent overview of various forms of (generalized) DTs and their applications in soliton theory.

In this note we propose a unified formulation of DTs encapsulating many of the examples in [6]. The key is the interpretation of DTs as gauge transformations involving gauge operators parametrized by (adjoint) eigenfunctions. The use of pseudo-differential symbols allows a formal inversion of differential operators and leads to a compact formalism to define the relevant gauge operators. They also lead to the connection with Sato's definition of the (multicomponent KP) hierarchy [7-11], so that compact formulas for the KP solutions generated by DTs can be established.

No proofs are given in this note. For the scalar KP hierarchy the formulation of the classical DT as a pseudo-differential gauge transformation and corresponding results for the modified KP are given and proven in [12,13]. Details and proofs for the DT and its iteration in context of the multicomponent KP hierarchy are given in [14]. Details and proofs for the adjoint and binary transformations for the KP hierarchy and the modified KP hierarchy are to be published elsewhere.

2. Motivation: the classical Darboux transformation as a gauge transformation

The Lax equation $L_t = [M, L]$ for the differential operators

$$L = L(u) = \partial^2 + u \quad , \quad M = M(u) = 4\partial^3 + 6u\partial + 3u_x \tag{2.1}$$

is known to imply the KdV equation

$$u_t = u_{xxx} + 6uu_x \tag{2.2}$$

for the potential u. It represents the compatibility condition of the spectral problem for the Schrödinger operator with the time evolution of the eigenfunction given by M, i.e.

$$[L\Phi] = \lambda\Phi \quad , \quad \Phi_t = [M\Phi] \tag{2.3}$$

(where $[Aa]$ denotes the action of the operator A on the function a).

Darboux's classical observation is that with an eigenfunction Φ any other eigenfunction χ solving $[L(u)\chi] = \mu\chi$ is mapped to an eigenfunction of a new Schrödinger operator $[L(\tilde{u})\tilde{\chi}] = \mu\tilde{\chi}$ by

$$\tilde{\chi} = \Phi(\Phi^{-1}\chi)_x \quad , \quad \tilde{u} = u + 2(ln(\Phi))_{xx} \quad . \tag{2.4}$$

We observe that the spectral parameter μ remains unchanged in this transformation. A direct computation also shows that, if $\chi_t = [M(u)\chi]$, then the new eigenfunction $\tilde{\chi}$ satisfies the time evolution $\tilde{\chi}_t = [M(\tilde{u})\tilde{\chi}]$ associated with the new potential $\tilde{u}$. Hence, via compatibility, one concludes that $\tilde{u}$ is a new solution of the KdV. This provides the following mechanism to generate exact solutions of the KdV: one needs a first (maybe trivial) solution u of the KdV. Then one solves the <u>linear</u> problems (2.3) to obtain a new solution $\tilde{u}$ of the nonlinear KdV.

As the eigenvalue problem is mapped to the new eigenvalue problem with the same spectral parameter, one expects that the transformed Schrödinger operator $\tilde{L} = L(\tilde{u})$ is obtained from the original operator $L = L(u)$ via a transformation of the type $\tilde{L} = TLT^{-1}$ with some suitable operator T. As T applied to an eigenfunction χ of L should produce an eigenfunction $\tilde{\chi}$ of $\tilde{L}$, it is natural to consider the operator

$$T = \Phi\partial\Phi^{-1} = \partial - \Phi_x\Phi^{-1} \quad . \tag{2.5}$$

We introduce a formal integration ∂^{-1} to invert (2.5) via $T^{-1} = \Phi\partial^{-1}\Phi^{-1}$. An elementary calculation shows that the operator identities

$$\begin{aligned} TL(u)T^{-1} \quad &= L(\tilde{u}) + \left(\frac{\Phi_{xx}+u\Phi}{\Phi}\right)_x \partial^{-1}\Phi^{-1} \\ TM(u)T^{-1} + T_tT^{-1} \ &= M(\tilde{u}) - (\Phi^{-1}\{\Phi_t - [M(u)\Phi]\})_x \, \Phi\partial^{-1}\Phi^{-1} \end{aligned} \tag{2.6}$$

are satisfied. With (2.3) these (formal) integro-differential operators reduce to the pair $L(\tilde{u}), M(\tilde{u})$ of differential operators, which represent the new Lax pair associated with the potential $\tilde{u}$.

To summarize: Darboux's classical transformation of the Lax pair (2.1) for the KdV may be understood as the gauge transformation

$$L \ \rightarrow \ \tilde{L} = TLT^{-1} \quad , \quad M \ \rightarrow \ \tilde{M} = TMT^{-1} + T_tT^{-1} \tag{2.7}$$

with the gauge operator (2.5), which is parametrized by an eigenfunction Φ of the Lax pair L, M.

3. Background

For a general discussion of DTs we consider Lax hierarchies $L_{t_A} = [M_A, L]$, where A is some index labeling the different flow parameters t_A of the hierarchy. The differential matrix operators M_A generating these flows are to satisfy the zero-curvature condition

$$M_{A,t_B} - M_{B,t_A} = [M_B, M_A] \tag{3.1}$$

for all pairs of indices A, B, so that the corresponding flows are compatible.

We consider gauge transformations

$$L \;\rightarrow\; \tilde{L} = \mathcal{T} L \mathcal{T}^{-1} \quad , \quad M_A \;\rightarrow\; \tilde{M}_A = \mathcal{T} M_A \mathcal{T}^{-1} + \mathcal{T}_{t_A} \mathcal{T}^{-1} \quad , \tag{3.2}$$

which preserve the zero-curvature condition as well as the Lax formulations. As motivated in the previous section, differential gauge operators $\mathcal{T}$ have to be considered, which need to be inverted by some suitable formalism. In order to give a rigorous meaning to such inverses, we regard differential operators as special elements of the larger algebra

$$g = \{ \sum_{-\infty < i \ll \infty} u_i \, \partial^i \, \} \tag{3.3}$$

of pseudo-differential symbols. The coefficients (multiplication operators) $u_i = u_i(x)$ are elements of some matrix algebra, $\partial = \partial_x$ is the differential operator, and formal integrations given by negative powers of ∂ are defined by the Leibniz rule

$$\partial^i u = \sum_{j=0}^{\infty} \frac{i(i-1)..(i-j+1)}{1 \times 2 \times .. \times j} \; \frac{d^j u}{dx^j} \; \partial^{i-j} \quad , \tag{3.4}$$

which for negative i leads to an infinite Laurent expansion in the symbol ∂. This, together with $\partial\partial^{-1} = \partial^{-1}\partial = 1$ yields a well defined associative algebraic structure on g. The transposition of such symbols is defined in the usual way by

$$(\sum_i u_i \partial^i)^* = \sum_i (-1)^i \partial^i u_i^\dagger \quad , \tag{3.5}$$

where $\dagger$ is to denote matrix transposition. We use the notation

$$P_{\geq k}(M) = \sum_{i \geq k} u_i \partial^i \quad , \quad P_{<k}(M) = \sum_{i<k} u_i \partial^i \tag{3.6}$$

for projections of a pseudo-differential symbol $M = \sum_i u_i \partial^i$ onto its corresponding differential orders. A further important projection is the residue of a pseudo-differential symbol given by $res(\sum_i u_i \partial^i) = u_{-1}$.

Functions will be identified with multiplication operators. Hence, for an operator $M \in g$ and a function Φ we will always use the notation $M\Phi$ for the operator, which is the composition of M with the multiplication operator given by Φ. If a differential operator M is to act on Φ, we will denote the resulting function by $[M\Phi]$. For a given differential operator M we consider eigenfunctions and adjoint eigenfunctions, which are defined as solutions of the linear evolution equations

$$\Phi_t = [M\Phi] \quad , \quad \Psi_t = -[M^*\Phi] \quad . \tag{3.7}$$

The letter Φ will be reserved for eigenfunctions throughout this paper, whereas Ψ will always denote an adjoint eigenfunction. We observe the important fact that associated with each pair Φ, Ψ there is a bi-linear potential $\Omega(\Psi, \Phi)$ integrating the "squared eigenfunction" $\Psi^\dagger\Phi$.

Lemma 1:

For any matrix differential operator M and any pair of (adjoint) eigenfunctions Φ and Ψ there exists a potential $\Omega(\Psi, \Phi)$ satisfying

$$\Omega(\Psi, \Phi)_x = \Psi^\dagger\Phi \quad , \quad \Omega(\Psi, \Phi)_t = res(\partial^{-1}\Psi^\dagger M\Phi\partial^{-1}) \quad . \tag{3.8}$$

This potential will play a crucial role in the (binary) DTs to be considered. It is defined up to an integration constant and may be regarded as a contour integration of the corresponding expressions in the (x, t)-plane. Solutions Φ, Ψ of the linear problems (3.7) may be either vectors (columns) or several vector solutions may be combined to matrix solutions. In the first case the potential Ω associated with such a pair is a scalar function, whereas in the latter case Ω is a matrix.

4. Darboux theorems

In this section we consider the linear problems (3.7) where

$$M = u_N\partial^N + u_{N-1}\partial^{N-1} + .. + u_1\partial + u_0 \tag{4.1}$$

is any matrix differential operator. We consider gauge transformations

$$\begin{array}{l} M \;\rightarrow\; \tilde{M} = \mathcal{T}M\mathcal{T}^{-1} + \mathcal{T}_t\mathcal{T}^{-1} \quad , \\ \Phi \;\rightarrow\; \tilde{\Phi} = [\![\mathcal{T}\Phi]\!] \quad , \quad \Psi \;\rightarrow\; \tilde{\Psi} = [\![\mathcal{T}^{-1*}\Psi]\!] \quad , \end{array} \tag{4.2}$$

which is given by some time-dependent operator $\mathcal{T}$. Expressions such as $[\![\mathcal{T}\Phi]\!]$ are to denote "application of the operator $\mathcal{T}$ to the function Φ". For a differential operator $\mathcal{T}$ this was introduced as $[\mathcal{T}\Phi]$. For operators with formal pseudo-differential parts, however, it is a priori not clear what this action is to be, and special care has to be taken to give a rigorous meaning to this operation. Let us assume that one has $[\![A(B\xi)]\!] = [\![(AB)\xi]\!]$, $[\![A\xi]\!]_t = [\![A_t\xi + A\xi_t]\!]$ for operators A, B and functions ξ. Then, formally, the new (adjoint) eigenfunctions $\tilde{\Phi}$, $\tilde{\Psi}$ again satisfy the scattering problems (3.7) with M replaced by $\tilde{M}$, as

$$\begin{aligned}\tilde{\Phi}_t &= [\![\mathcal{T}_t\Phi + \mathcal{T}\Phi_t]\!] = [\![(\mathcal{T}_t\mathcal{T}^{-1} + \mathcal{T}M\mathcal{T}^{-1})\mathcal{T}\Phi]\!] = [\![\tilde{M}\tilde{\Phi}]\!] \quad , \\ \tilde{\Psi}_t &= [\![\mathcal{T}_t^{-1*}\Psi + \mathcal{T}^{-1*}\Psi_t]\!] = -[\![(\mathcal{T}^{-1*}\mathcal{T}_t^* + \mathcal{T}^{-1*}M\mathcal{T}^*)\mathcal{T}^{-1*}\Psi]\!] \\ &= -[\![\tilde{M}^*\tilde{\Psi}]\!] \quad .\end{aligned} \tag{4.3}$$

This analysis, however, holds rigorously only if all operators involved were purely differential operators, so that $[\![.]\!] = [.]$ is well-defined. As already seen in Section 2 for the Schrödinger operator, the inversion of differential operators leads naturally to formal integration symbols, and the definition of the gauge transformed (adjoint) eigenfunctions $\tilde{\Phi}$ and $\tilde{\Psi}$ involving $[\![.]\!]$ will have to be given rigorously for each case to be considered in the following.

In particular, we have to ensure that the gauge transformed operator $\tilde{M} = \mathcal{T}M\mathcal{T}^{-1} + \mathcal{T}_t\mathcal{T}^{-1}$ is again a differential operator. This is the main restriction on the transformation operators $\mathcal{T}$ to be considered. In general, we think of $\mathcal{T}$ as a pseudo-differential symbol, which is to be inverted in the pseudo-differential algebra g given by (3.3). Hence, in general, if we start with a differential operator M, the resulting operator $\tilde{M}$ will not be a differential operator any more, but it will have a part defined by formal integration symbols. In order to ensure that no negative differential orders turn up in $\tilde{M}$, we impose the condition that the negative orders of the term $\mathcal{T}M\mathcal{T}^{-1}$ are cancelled by the additional term $\mathcal{T}_t\mathcal{T}^{-1}$. Hence, we restrict our considerations to gauge operators satisfying the following "gauge equation"

$$\mathcal{T}_t = -P_{<0}(\mathcal{T}M\mathcal{T}^{-1})\mathcal{T} \quad , \tag{4.4}$$

which –for given M– may be considered as a non-linear evolution equation for $\mathcal{T}$. The gauge transformation $M \to \tilde{M}$ now can be written as

$$M \quad \to \quad \tilde{M} = P_{\geq 0}(\mathcal{T}M\mathcal{T}^{-1}) \quad . \tag{4.5}$$

In this construction the use of pseudo-differential symbols becomes a convenient technical device to define the formal operator $\mathcal{T}M\mathcal{T}^{-1}$. The gauge transformed differential operator $\tilde{M}$ is then given as the projection of the formal integro-differential operator $\mathcal{T}M\mathcal{T}^{-1}$ to its purely differential part.

We shall now consider concrete realizations of suitable gauge operators. The crucial observation behind the following DTs is the fact that the <u>nonlinear</u> evolution equation (4.4) can be <u>linearized</u>, i.e. there exist realizations of T in terms of objects satisfying linear equations. In particular, eigenfunctions and adjoint eigenfunctions associated with the differential operator M can be used to parametrize solutions of the gauge equation (4.4).

Theorem 1: (The "elementary" transformations)

Let Φ, Ψ be solutions of (3.7). Then the following operators

a) $T(\Phi) = \Phi\partial\Phi^{-1}$ (with inverse $T^{-1} = \Phi\partial^{-1}\Phi^{-1}$),

b) $T^{(adj)}(\Psi) = \Psi^{-1\dagger}\partial^{-1}\Psi^{\dagger}$ (with inverse $(T^{(adj)})^{-1} = \Psi^{-1\dagger}\partial\Psi^{\dagger}$),

c) $T^{(bin)}(\Phi,\Psi) = 1 - \Phi\Omega(\Psi,\Phi)^{-1}\partial^{-1}\Psi^{\dagger}$

(with inverse $(T^{(bin)})^{-1} = 1 + \Phi\partial^{-1}\Omega(\Psi,\Phi)^{-1}\Psi^{\dagger}$)

satisfy equation (4.4).

Here, for a) and b) one needs to use invertible matrix solutions Φ and Ψ, whereas c) holds both for matrix or vector solutions of (3.7). As mentioned before, in the latter case the potential $\Omega(\Psi,\Phi)$ of Lemma 1 is a scalar.

Case a) is to be called a DT, as according to Section 2 this gauge operator is connected with Darboux's classical results for the Schrödinger operator. For reasons to be explained soon, case b) shall be called the "adjoint DT". Case c) will turn out to be a composition of the first two cases, following [6] it is to be called a "binary DT".

A crucial feature of DTs is the fact that they provide a mechanism to transform (adjoint) eigenfunctions to new (adjoint) eigenfunctions for the transformed differential operator $\tilde{M}$. In principle, the transformation rules (4.2) provide the new solutions of the linear problems. However, the transformation operators and their inverses involve the formal integration ∂^{-1} and cannot be applied directly to (adjoint) eigenfunctions. It turns out that formal integrations have to be replaced by the potential Ω of Lemma 1. The following theorem provides the correct interpretation of the formal transformations (4.2). We note that here as well as in the following Theorems 3-5 the symbol χ is reserved for eigenfunctions, whereas η will always refer to an adjoint eigenfunction:

Theorem 2:

Let Φ,Ψ be the (adjoint) eigenfunctions triggering the transformations of Theorem 1. Let χ and η be a further pair of (adjoint) eigenfunctions. Then

a) $\tilde{\chi}{=}\Phi(\Phi^{-1}\chi)_x$, $\tilde{\eta}{=}-\Phi^{-1\dagger}\Omega(\eta,\Phi)^{\dagger}$,

b) $\tilde{\chi}{=}\Psi^{-1\dagger}\Omega(\Psi,\chi)$, $\tilde{\eta}{=}-\Psi(\Psi^{-1}\eta)_x$,

c) $\tilde{\chi}{=}\chi - \Phi\Omega(\Psi,\Phi)^{-1}\Omega(\Psi,\chi)$, $\tilde{\eta}{=}\eta - \Psi\Omega(\Psi,\Phi)^{-1\dagger}\Omega(\eta,\Phi)^{\dagger}$,

solve $\tilde{\chi}_t = [\tilde{M}\tilde{\chi}]$, $\tilde{\eta}_t = -[\tilde{M}^*\tilde{\chi}]$ with $\tilde{M} = P_{\geq 0}(\mathcal{T}M\mathcal{T}^{-1})$.

We note that a superposition principle holds for the potentials $\Omega(\eta,\chi)$ of Lemma 1 associated with a pair of (adjoint) eigenfunctions. A straightforward calculation shows that

$$
\begin{aligned}
&\text{a)}\ \ \tilde{\Omega}(\tilde{\eta},\tilde{\chi}) = \Omega(\eta,\chi) - \Omega(\eta,\Phi)\Phi^{-1}\chi \quad , \\
&\text{b)}\ \ \tilde{\Omega}(\tilde{\eta},\tilde{\chi}) = \Omega(\eta,\chi) - \eta^{\dagger}\Psi^{-1\dagger}\Omega(\Psi,\chi) \quad , \\
&\text{c)}\ \ \tilde{\Omega}(\tilde{\eta},\tilde{\chi}) = \Omega(\eta,\chi) - \Omega(\eta,\Phi)\Omega(\Psi,\Phi)^{-1}\Omega(\Psi,\chi)
\end{aligned}
\tag{4.6}
$$

are potentials for the new pair $\tilde{\eta},\tilde{\chi}$ satisfying

$$
\tilde{\Omega}(\tilde{\eta},\tilde{\chi})_x = \tilde{\eta}^{\dagger}\tilde{\chi} \quad , \quad \tilde{\Omega}(\tilde{\eta},\tilde{\chi})_t = res(\partial^{-1}\tilde{\eta}^{\dagger}\tilde{M}\tilde{\chi}\partial^{-1}) \ . \tag{4.7}
$$

The adjoint DT b) is in fact a direct consequence of the original DT a) via transposition of the operators, which swaps the roles of eigenfunctions and adjoint eigenfunctions. If we start with an operator M equipped with an adjoint eigenfunction Ψ, we may consider the adjoint operator $M' = -M^*$, which is equipped with the eigenfunction $\Phi' = \Psi$. Now a DT with Φ' leads to an operator $\tilde{M}'$. A final transposition to $\tilde{M} = -\tilde{M}'^*$ yields $\tilde{M} = P_{\geq 0}(TM^*T^{-1})^* = P_{\geq 0}(T^{-1*}MT^*)$, where $T = T(\Phi') = T(\Psi) = \Psi\partial\Psi^{-1}$ gives the DT from M' to $\tilde{M}'$. Hence,

$$
\mathcal{T} = T(\Psi)^{-1*} = (\Psi\partial^{-1}\Psi^{-1})^* = -\Psi^{-1\dagger}\partial^{-1}\Psi^{\dagger} \tag{4.8}
$$

is the resulting gauge operator mapping $M \to \tilde{M} = P_{\geq 0}(\mathcal{T}M\mathcal{T}^{-1})$. Up to the irrelevant minus sign, this is the adjoint DT given by case b) of Theorem 1. The results of Theorem 2.b) are obtained by introducing further (adjoint) eigenfunctions for M and tracing the results of Theorem 2.a) in the diagram

$$
\begin{tikzcd}
\boxed{M,\Psi} \arrow[rr, dashed, "T^{(adj)}(\Psi)"] \arrow[d, "\begin{array}{rcl} M' &=& -M^* \\ \Phi' &=& \Psi' \end{array}"'] & & \boxed{\tilde{M}} \\
\boxed{M',\Phi'} \arrow[rr, "T(\Phi')"] & & \boxed{\tilde{M}'} \arrow[u, "\tilde{M} = -\tilde{M}'^*"']
\end{tikzcd}
$$

<u>*Diagram 1*</u>

Hence, it is seen that the adjoint transformation is indeed the original DT conjugated with transposition of the operators. We remark that one may also regard the adjoint transformation as an inverse DT. We observe that according to Theorem 2.b) under an adjoint DT $M \to \tilde{M}$ given by $T^{(adj)}(\Psi)$ the transformed operator $\tilde{M}$ inherits an eigenfunction $\tilde{\chi} = \Psi^{-1\dagger}$. This is the image of the trivial eigenfunction $\chi = 0$ for M, by choosing the potential $\Omega(\Psi,\chi) = \Omega(\Psi,0) = 1$. If we trigger a DT of $\tilde{M}$ with this $\tilde{\chi}$, then the corresponding gauge operator is

$$
T(\tilde{\chi}) \ = \ \tilde{\chi}\partial\tilde{\chi}^{-1} \quad = \ \Psi^{-1\dagger}\partial\Psi^{\dagger} \ = \ T^{(adj)}(\Psi)^{-1} \ . \tag{4.9}
$$

In this sense the adjoint transformation is the inverse of a DT, which is triggered by the eigenfunction $\tilde{\chi} = \Psi^{-1\dagger}$ of the transformed operator.

The binary DT of Theorem 1c), 2c) is a composition of the cases a) and b). We start with the operator M, equipped with the eigenfunction Φ and the adjoint eigenfunction Ψ. We first perform a DT triggered by Φ, which results in an operator $M^{(1)}$, say. The adjoint eigenfunction Ψ is mapped to the adjoint eigenfunction $\Psi^{(1)} = -\Phi^{-1\dagger}\Omega(\Psi,\Phi)^\dagger$ by this DT. A subsequent adjoint DT using $\Psi^{(1)}$ maps $M^{(1)}$ to the final operator $\tilde{M}$, say. The composition of these two transformations is given by the gauge operator

$$\begin{aligned} &T^{(adj)}(\Psi^{(1)})\; T(\Phi) = (\Psi^{(1)})^{-1\dagger}\partial^{-1}\Psi^{(1)\dagger} \quad \Phi\partial^{-1}\Phi^{-1} \\ &= \Phi\Omega(\Psi,\Phi)^{-1}\partial^{-1}\Omega(\Psi,\Phi)\partial\Phi^{-1} = 1 - \Phi\Omega(\Psi,\Phi)^{-1}\partial^{-1}\Psi^\dagger \quad , \end{aligned} \tag{4.10}$$

which represents the binary transformation of Theorem 1c). We may also start from M and first perform an adjoint DT with Ψ, leading to a transformed operator $M^{(2)}$, say. The eigenfunction Φ is mapped to the eigenfunction $\Phi^{(2)} = \Psi^{-1\dagger}\Omega(\Psi,\Phi)$ for $M^{(2)}$ by this transformation. A subsequent DT triggered by $\Phi^{(2)}$ leads to the composed gauge operator

$$\begin{aligned} &T(\Phi^{(2)})\; T^{(adj)}(\Psi) = \Phi^{(2)}\partial(\Phi^{(2)})^{-1} \quad \Psi^{-1\dagger}\partial^{-1}\Psi^\dagger \\ &= \Psi^{-1\dagger}\Omega(\Psi,\Phi)\partial\Omega(\Psi,\Phi)^{-1}\partial^{-1}\Psi^\dagger = 1 - \Phi\Omega(\Psi,\Phi)^{-1}\partial^{-1}\Psi^\dagger \quad , \end{aligned} \tag{4.11}$$

i.e. again to the binary DT. Hence, the binary transformation may be regarded as the map $M \to \tilde{M}$ resulting from the following commuting diagram:

$$\begin{tikzcd} & \boxed{M,\Phi,\Psi} \arrow[dl, "T(\Phi)"'] \arrow[dr, "T^{(adj)}(\Psi)"] & \\ \boxed{M^{(1)},\Psi^{(1)}} \arrow[dr, "T^{(adj)}(\Psi^{(1)})"'] & & \boxed{M^{(2)},\Phi^{(2)}} \arrow[dl, "T(\Phi^{(2)})"] \\ & \boxed{\tilde{M}} & \end{tikzcd}$$

Diagram 2

In this derivation we have to assume that the pair Φ, Ψ triggering the binary transformation is given by invertible matrices, so that the intermediate (adjoint) DTs can be carried out. However, the final formula for the binary gauge operator also makes sense for vector solutions Φ and Ψ and an associated scalar potential $\Omega(\Psi,\Phi)$.

We remark that for a binary DT $M \to \tilde{M}$ triggered by the pair Φ, Ψ the gauge transformed operator automatically is equipped with the pair

$$\tilde{\chi} = \Phi\Omega(\Psi,\Phi)^{-1} \quad , \quad \tilde{\eta} = \Psi\Omega(\Psi,\Phi)^{-1\dagger} \quad , \tag{4.12}$$

of (adjoint) eigenfunctions, which are the images of the trivial (adjoint) eigenfunctions $\chi = 0$, $\eta = 0$ with the choice $\Omega(\Psi, \chi) = \Omega(\eta, \Phi) = -1$. Fixing $\Omega(\eta, \chi) = 0$ one finds the new potential $\tilde{\Omega}(\tilde{\eta}, \tilde{\chi}) = -\Omega(\Psi, \Phi)^{-1}$ according to Theorem 2c). The binary gauge operator triggered by this new pair is

$$\begin{aligned} T^{(bin)}(\tilde{\chi}, \tilde{\eta}) &= 1 - \tilde{\chi}\tilde{\Omega}(\tilde{\eta}, \tilde{\eta})^{-1}\partial^{-1}\tilde{\eta}^{\dagger} \\ &= 1 + \Phi\partial^{-1}\Omega(\Psi, \Phi)^{-1}\Psi^{\dagger} = (T^{(bin)}(\Phi, \Psi))^{-1} \ , \end{aligned} \tag{4.13}$$

so that a further binary DT triggered by $\tilde{\chi}, \tilde{\eta}$ maps $\tilde{M}$ back to the original operator M.

So far the (adjoint) DTs of Theorem 1.a),1.b) were used to derive the binary DT as a composition of the elementary (adjoint) DTs. Further gauge operators satisfying (4.4) can be constructed from further compositions of the elementary transformations. As an example, we may start with an operator equipped with a collection of matrix eigenfunctions $\Phi_1, .., \Phi_n$. Performing a first DT with Φ_1, say, we may use Theorem 2.a) to generate eigenfunctions $\tilde{\Phi}_2, .., \tilde{\Phi}_n$ for the gauge transformed operator. The eigenfunction Φ_1 used to trigger the transformation is lost, as it is mapped to zero. After this first transformation a second DT triggered by $\tilde{\Phi}_2$ is performed, leading again to eigenfunctions which are the images of $\tilde{\Phi}_3, .., \tilde{\Phi}_n$. Repeating this process one can perform n DTs until all the eigenfunctions are "used up". The resulting gauge operator is an nth order differential operator given as the composition of the first order operators triggering the individual DTs. It turns out that the final result does not depend on the ordering, in which the eigenfunctions are used for the DTs, and the final operator can be written in a compact way in terms of "Wronskian expressions" of the eigenfunctions.

Theorem 3: (Iteration of the DT)

Let $\Phi_1, .., \Phi_n$ be a collection of invertible matrix solutions of $\Phi_t = [M\Phi]$. Let $a_0, .., a_{n-1}$ and $b_1, .., b_n$ be the matrix functions determined as the solution of the linear algebraic system

$$\begin{gathered} \sum_{i=0}^{n} a_i \frac{d^i\Phi_j}{dx^i} = 0 \quad , \quad j = 1, .., n \quad , \quad a_n = 1 \ , \\ \sum_{j=1}^{n} \frac{d^i\Phi_j}{dx^i} b_j = \begin{cases} 0 & \text{for } i = 0, .., n-2 \ , \\ 1 & \text{for } i = n-1 \ . \end{cases} \end{gathered} \tag{4.14}$$

Then the operator

$$T_n = \sum_{i=0}^{n} a_i \partial^i \qquad \left(\text{ with inverse } \ T_n^{-1} = \sum_{j=1}^{n} \Phi_j \partial^{-1} b_j \ \right) \tag{4.15}$$

satisfies (4.4). A pair of (adjoint) eigenfunctions χ and η of M is transformed into the new pair of (adjoint) eigenfunctions

$$\tilde{\chi} = \sum_{i=0}^{n} a_i \frac{d^i \chi}{dx^i} \quad , \quad \tilde{\eta} = -\sum_{j=1}^{n} b_j^\dagger \Omega(\eta, \Phi_j)^\dagger \tag{4.16}$$

for the transformed operator $\tilde{M} = P_{\geq 0}(T_n M T_n^{-1})$.

For scalar operators and scalar eigenfunctions the solutions of the linear system (4.14) can be given by Cramer's rule in terms of Wronskian type determinants, and T_n, T_n^{-1} can be rewritten by compact Wronskian formulas [14]. In particular, the transformed eigenfunction is given by

$$\tilde{\chi} = \frac{\mathcal{W}(\Phi_1, .., \Phi_n, \chi)}{\mathcal{W}(\Phi_1, .., \Phi_n)} \quad , \tag{4.17}$$

where $\mathcal{W}$ is the Wronskian determinant of the indicated arguments.

A similar construction holds for the iteration of the adjoint DT. Starting with a collection $\Psi_1, .., \Psi_n$ of adjoint eigenfunctions for M, the composition of the n-fold adjoint transformation is given as follows.

Theorem 4: (Iteration of the adjoint DT)

Let $\Psi_1, .., \Psi_n$ be a collection of invertible matrix solutions of $\Psi_t = -[M^* \Psi]$. Let $a_0, .., a_{n-1}$ and $b_1, .., b_n$ be the matrix functions determined as the solution of the linear algebraic system

$$\begin{aligned} \sum_{i=0}^{n} a_i \frac{d^i \Psi_j}{dx^i} &= 0 \quad , \quad j = 1, .., n \quad , \quad a_n = 1 \; , \\ \sum_{j=1}^{n} \frac{d^i \Psi_j}{dx^i} b_j &= \begin{cases} 0 & \text{for } i = 0, .., n-2 \; , \\ 1 & \text{for } i = n-1 \; . \end{cases} \end{aligned} \tag{4.18}$$

Then the operator

$$T_n^{(adj)} = (-1)^{n+1} \sum_{j=1}^{n} b_j^\dagger \partial^{-1} \Psi_j^\dagger \tag{4.19}$$

has the inverse

$$(T_n^{(adj)})^{-1} = (-1)^n (\sum_{i=0}^{n} a_i \partial^i)^* \tag{4.20}$$

and satisfies (4.4). A pair of (adjoint) eigenfunctions χ and η of M is transformed into the new pair of (adjoint) eigenfunctions

$$\tilde{\chi} = (-1)^{n+1} \sum_{j=1}^{n} b_j^\dagger \Omega(\Psi_j, \chi) \quad , \quad \tilde{\eta} = (-1)^n \sum_{i=0}^{n} a_i \frac{d^i \eta}{dx^i} \tag{4.21}$$

for the transformed operator $\tilde{M} = P_{\geq 0}(T_n^{(adj)} M (T_n^{(adj)})^{-1})$.

In a similar way iterations of the binary DT can be formulated. As the transformation involves the potential $\Omega(\Psi,\Phi)$ of the triggering pair Φ,Ψ explicitly, it is useful to have the potentials associated with transformed pairs given by the superposition formula (4.6.c). The result is as follows.

Theorem 5: (Iteration of the binary DT)

Let $\Phi_1,..,\Phi_n,\Psi_1,..,\Psi_n$ be a collection of vector (or matrix) solutions of (3.7), let $\Omega_{ij}=\Omega(\Psi_i,\Phi_j)$ be the matrix of associated potentials. Let $a_1,..,a_n$ and $b_1,..,b_n$ be the vector (or matrix) functions determined as the solution of the linear algebraic system

$$\sum_{j=1}^{n} a_j\Omega_{ji}=\Phi_i \quad , \quad \sum_{j=1}^{n} b_j\Omega_{ij}^{\dagger}=\Psi_i \quad , \quad i=1,..,n\ . \tag{4.22}$$

Then the operator

$$T_n^{(bin)}=1-\sum_{i=1}^{n} a_i\partial^{-1}\Psi_i^{\dagger} \tag{4.23}$$

has the inverse

$$(T_n^{(bin)})^{-1}=1+\sum_{i=1}^{n}\Phi_i\partial^{-1}b_i^{\dagger} \tag{4.24}$$

and satisfies (4.4). A pair of (adjoint) eigenfunctions χ and η of M is transformed into the new pair of (adjoint) eigenfunctions

$$\tilde\chi=\chi-\sum_{i=1}^{n} a_i\Omega(\Psi_i,\chi) \quad , \quad \tilde\eta=\eta-\sum_{i=1}^{n} b_i\Omega(\eta,\Phi_i)^{\dagger} \tag{4.25}$$

for the transformed operator $\tilde M=P_{\geq 0}(T_n^{(bin)}M(T_n^{(bin)})^{-1})$.

We remark that by choosing trivial (adjoint) eigenfunctions $\chi=\eta=0$ and $\Omega(\Psi_i,\chi)=\Omega(\eta,\Phi_i)=\delta_{ij}$ it is readily seen, that the coefficients a_j and b_j are eigenfunctions and adjoint eigenfunctions of $\tilde M$. A further transformation triggered by the a_j and b_j will map $\tilde M$ back to the original operator M.

5. Applications: Solution Formulas for the KP Hierarchy

A convenient parametrization of zero-curvature matrices M_A, M_B satisfying (3.1) is given by Sato's construction [7-11] of the (multicomponent) KP hierarchy. One introduces the dressing operator

$$W=1+w_1\partial^{-1}+w_2\partial^{-2}+... \tag{5.1}$$

and its inverse

$$W^{-1}=1-w_1\partial^{-1}+(w_1^2-w_2)\partial^{-2}+... \tag{5.2}$$

and imposes the following hierarchy of evolution equations

$$W_{t_A}=-P_{<0}(WAW^{-1})W \quad . \tag{5.3}$$

Here A is taken from some set $\mathcal{A}$ of commuting constant matrix operators, and with each A a different flow-parameter (time) t_A is associated. The infinite set of nonlinear partial differential equations (5.3) for the matrix coefficients $w_1, w_2, ..$ is called the multicomponent KP hierarchy (associated with the collection $\mathcal{A}$ of "undressed" (or "bare") operators). Depending on the underlying matrix algebra used for $\mathcal{A}$ and the coefficients w_i of W, various nonlinear equations for finitely many of the fields w_i can be extracted from the KP hierarchy. These include the KP equation itself for the scalar case as well as the Davey-Stewartson equation for 2×2 matrices or the N-wave system for $N \times N$ matrices [15].

One defines the hierarchy of differential operators

$$M_A = P_{\geq 0}(WAW^{-1}) \quad , \tag{5.4}$$

which are labeled by the bare operators $A \in \mathcal{A}$. It is readily checked that for all pairs $A, B \in \mathcal{A}$ the zero-curvature condition (3.1) is satisfied, which is a consequence of the commutativity of the bare operators. Further, for any fixed operator $C \in \mathcal{A}$, the pseudo-differential Lax operator $L = WCW^{-1}$ satisfies the hierarchy of commuting Lax equations

$$L_{t_A} = [M_A, L] \quad . \tag{5.5}$$

The previous DTs can be applied to the differential operators M_A and lead to solutions formulas of the KP hierarchy. We observe that (5.4) defines a parametrization of the zero-curvature operators in terms of the coefficients w_i of the dressing operator. This parametrization is preserved under (adjoint,binary) DTs, as they are generated by transformations of the dressing operator W itself:

Theorem 6: (The Darboux Transformations for the dressing operator)
Let W solve the KP hierarchy (5.3), let Φ and Ψ be (adjoint) eigenfunctions solving

$$\Phi_{t_A} = [M_A \Phi] \;, \Psi_{t_A} = -[M_A^* \Psi] \;. \tag{5.6}$$

Then

$$\begin{aligned} &\text{a)} \;\; \tilde{W} = T(\Phi) W \partial^{-1} \quad , \\ &\text{b)} \;\; \tilde{W} = T^{(adj)}(\Psi) W \partial \quad , \\ &\text{c)} \;\; \tilde{W} = T^{(bin)}(\Phi, \Psi) W \quad , \end{aligned} \tag{5.7}$$

are again solutions of the KP hierarchy (5.3).

These statements are immediate consequences of the gauge equation (4.4) with $t = t_A$ satisfied by the gauge operators. The multiplication with ∂^{-1} and ∂ in the cases a) and b) is to renormalize the leading coefficient of $\tilde{W}$ to 1. The transformation

$$M_A = P_{\geq 0}(WAW^{-1}) \;\longrightarrow\; \tilde{M}_A = P_{\geq 0}(\tilde{W} A \tilde{W}^{-1}) \tag{5.8}$$

is given by $\tilde{M}_A = P_{\geq 0}(\mathcal{T} M_A \mathcal{T}^{-1})$, so that Theorem 6 indeed generates the DTs for the differential operators given by (5.4). Hence, the parametrization of these operators in terms of the coefficients w_i is left invariant by the (adjoint,binary) DTs.

The iterations of the elementary transformations lead to solutions

$$\begin{aligned} &\text{a)} \ \ \tilde{W} = T_n W \partial^{-n} \quad , \\ &\text{b)} \ \ \tilde{W} = T_n^{(adj)} W \partial^{n} \quad , \\ &\text{c)} \ \ \tilde{W} = T_n^{(bin)} W \quad , \end{aligned} \tag{5.9}$$

where the gauge operators $T_n, T_n^{(adj)}, T_n^{(bin)}$ of Theorems 3-5 are given in terms of collections of (adjoint) eigenfunctions.

To summarize, a first solution W of the nonlinear KP hierarchy (5.3) and solutions of the linear problems (5.6) provide a new solution of the KP via (5.9) . In particular, starting with the trivial solution $W = 1$, one obtains the compact solution formula

$$a) \ \ \tilde{W} = T_n \partial^{-n} \quad , \quad b) \ \ \tilde{W} = T_n^{(adj)} \partial^{n} \quad , \quad c) \ \ \tilde{W} = T_n^{(bin)} \tag{5.10}$$

given by the gauge operators of Theorems 3-5. In this case the (adjoint) eigenfunctions defining the gauge operators are given as solutions of the <u>constant coefficient</u> PDEs $\Phi_{t_A} = [A\Phi]$ and $\Psi_{t_A} = -[A^*\Psi]$.

We finally remark that the DTs can also be applied to reductions of the general KP hierarchy. One type of reduction is given by dressing operators satisfying the constraint

$$W^* S W = S \quad , \tag{5.11}$$

where S is any given constant symmetric matrix. This constraint is left invariant by those evolutions (5.3), which are generated by bare operators A satisfying

$$A^* S = -SA \quad . \tag{5.12}$$

The associated differential operators $M_A = P_{\geq 0}(WAW^{-1})$ satisfy

$$M_A^* S = -S M_A \ . \tag{5.13}$$

In general, neither the DT nor the adjoint DT will leave the constraint (5.13) invariant. However, for $\Psi = S\Phi$ the binary transformation operator $T^{(bin)} = T^{(bin)}(\Phi, S\Phi)$ satisfies

$$(T^{(bin)})^* S T^{(bin)} = S \quad , \tag{5.14}$$

so that $\tilde{W} = T^{(bin)}W$ and $\tilde{M}_A = P_{\geq 0}(T^{(bin)}M_A(T^{(bin)})^{-1})$ again satisfy the constraints (5.11) and (5.13). Hence, starting from the trivial solution $W = 1$ and iterating the binary transformation, one obtains exact solutions $\tilde{W} = T_n^{(bin)}$ of the KP hierarchy subject to the constraint (5.11). The binary gauge operator is parametrized by a set of eigenfunctions $\Phi_1, .., \Phi_n$ only, as the adjoint eigenfunctions Ψ_i have to be chosen as $\Psi_i = S\Phi_i$, $i = 1, .., n$, with associated potentials $\Omega(S\Phi_i, \Phi_j) = \Omega(S\Phi_j, \Phi_i)$.

Other reductions –connected with 1+1-dimensional soliton equations– are given by constraints of the type $P_{<0}(WCW^{-1}) = 0$ with some constant matrix $C \in \mathcal{A}$. The differential Lax operator $L = WCW^{-1}$ is parametrized by the fields $w_1, .., w_N$, if N is the order of the constant operator C. In this case the KP hierarchy (5.3) (or the Lax hierarchy (5.5), respectively) reduces to the hierarchy of isospectral 1+1-dimensional equations for $w_1, .., w_N$ associated with the Lax operator L. In this situation the (adjoint) eigenfunctions have to satisfy the spectral equations

$$[L\Phi] = \lambda\Phi \quad , \quad [L^*\Psi] = \mu\Psi \quad . \tag{5.15}$$

These conditions guarantee that for the (adjoint) DT $L \to \tilde{L} = \mathcal{T}L\mathcal{T}^{-1}$ the resulting $\tilde{L}$ is again a differential operator. Due to (5.15) the potential $\Omega(\Psi, \Phi)$ can be given explicitly by the formula

$$\Omega(\Psi, \Phi) = \frac{res(\partial^{-1}\Psi^\dagger L\Phi\partial^{-1})}{\lambda - \mu} \quad . \tag{5.16}$$

If we use this particular potential for the binary transformation, then also the binary case produces differential operators $\tilde{L}$. Hence, exact solutions of the 1+1-dimensional Lax equations (5.5) can be obtained from (adjoint,binary) DTs by using (adjoint) eigenfunctions satisfying the spectral problems (5.15).

6. Concluding remarks

All results presented here can be transformed into DTs for the modified multicomponent KP hierarchy. For the modified case the differential operators of the linear scattering problems are characterized by that fact that no zero order term is present. As a consequence, these scattering problems admit trivial (constant) eigenfunctions. One observes that the gauge operators

$$\tau(\Phi) = \Phi^{-1} \quad , \quad \tau^{(adj)}(\Psi) = \partial^{-1}\Psi^{-1\dagger} \quad , \tag{6.1}$$

satisfy the gauge equation $\mathcal{T}_t = -P_{<1}(\mathcal{T}M\mathcal{T}^{-1})\mathcal{T}$, where in contrast to (4.4) the zero order term of $\mathcal{T}M\mathcal{T}^{-1}$ is included. Gauge transformations $M \longrightarrow \tilde{M} = \mathcal{T}M\mathcal{T}^{-1} + \mathcal{T}_t\mathcal{T}^{-1} = P_{\geq 1}(\mathcal{T}M\mathcal{T}^{-1})$ with these operators yield differential operators with no zero order term and may be regarded as Miura transformations between the KP and the modified KP hierarchy. Compositions of the (adjoint,binary) DTs of Theorem 1 with these transformations lead to (adjoint,binary) DTs of the modified KP hierarchy. Thus, solution formulas such as (5.9)/(5.10) are obtained for the scalar modified KP equation and its multicomponent versions, which include the Ishimori system or the (2+1)-dimensional sine-Gordon system recently introduced by Konopelchenko and Rogers [16,17].

References:

[1] G. Darboux, *C.R. Acad. Sci. Paris* **94**, 1456 (1882).

[2] M.M. Crum, *Quart. J. Math.* **6**, 121 (1955).

[3] M. Wadati, H. Sanuki, K. Konno, *Prog. Theor. Phys.* **53**, 419 (1975).

[4] H. Wahlquist, in *Bäcklund Transformations*, R. Miura (ed.), Lect. Notes Math. Vol. **515**, Springer, Berlin (1976).

[5] M. Boiti, J.J-P. Leon, L. Martina, F. Pempinelli, *Phys. Lett.* **132A**, 432 (1988).

[6] V.B. Matveev, M.A. Salle, *Darboux Transformations and Solitons*, Springer Series in Nonlinear Dynamics, Springer, Berlin (1991).

[7] M. Sato, Y. Sato, *Lecture Notes in Num. Appl. Anal.* **5**, 259 (1982).

[8] M. Sato, Y. Sato, in *Nonlinear Partial Differential Equations in Applied Science*, H. Fujita, P.D. Lax and G. Strang (eds.), Kinokuniya/North Holland, Tokyo (1983).

[9] E. Date, M. Jimbo, M. Kashiwara, Miwa, in *Nonlinear Integrable Systems - Classical Theory and Quantum Theory*, M. Jimbo and T. Miwa (eds.), World Scientific, Singapore (1983).

[10] M. Jimbo, T. Miwa, *Publ. RIMS*, Kyoto Univ., **19**, 943 (1983).

[11] Y. Ohta, J. Satsuma, D. Takahashi, T. Tokihiro, *Progr. Theor. Phys. Suppl.* **94**, 210 (1988).

[12] W. Oevel, C. Rogers, *Gauge Transformations and Reciprocal Links in 2+1 Dimensions*, to appear in *Reviews in Mathematical Physics*.

[13] B.G. Konopelchenko, W. Oevel, *An r-Matrix Approach to Nonstandard Classes of Integrable Equations*, Mathematical Sciences Report A 149, Loughborough University (1991).

[14] W. Oevel, *Darboux Theorems and Wronskian Formulas For Integrable Systems I: Constrained KP Flows*, Mathematical Sciences Report A 172, Loughborough University (1991).

[15] B.G. Konopelchenko, W. Oevel, *Matrix Sato Theory and Integrable Equations in 2+1 Dimensions*, in the *Proceedings of the 7th Workshop on Nonlinear Evolution Equations and Dynamical Systems (NEEDS91)*, World Scientific, Singapore (1992).

[16] B.G. Konopelchenko, C. Rogers, *Phys. Lett.* **158A**, 391 (1991).

[17] B.G. Konopelchenko, W. Schief, C. Rogers, *A 2+1 dimensional sine-Gordon system: its auto-Baecklund transformation*, preprint (1992).